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VECTOR-DIAGRAMS OF OSCILLATING-CURRENT CIRCUITS.

By A. E. Kennelly.

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Although, as the appended bibliography indicates, the analysis of oscillating-current circuit phenomena has received much attention, and has attained a considerable degree of development, yet the methods of vector-diagrams do not seem to have been applied to it. It is believed that these vector-diagram methods offer marked advantages to the student, and particularly to students of electrical engineering who have already become familiar with the use of vector-diagrams in connection with alternating-current circuits.

Definition.

An oscillating-current circuit may be defined as a circuit which, in undergoing a change of energy, carries a current that oscillates periodically about an ultimate value. In such a circuit not only the electric current, but also the emf. quantity, power, and energy oscillate freely. For brevity, the term oscillating current may be designated by the letters *o. c.* In practice, an o. c. circuit comprises a condenser as an essential element, the energy oscillating, or tending to oscillate, between the electric and magnetic types, at a frequency determined solely by the constants of the circuit. The oscillations decay in amplitude, energy being expelled irrecoverably from the circuit, during the process, by a regular sequence of dwindling impulses. These impulses may be either of the joulean or *hertzian* type, or of both combined.

Simple Resistanceless Oscillating-Current Circuits.

The simplest type of o. c. circuit comprises a condenser AB, Figure 1, of capacity or permittance c farads, inserted in a circuit of negligible resistance, containing a total inductance of l henrys. Let n be the free-oscillation frequency of the circuit, in cycles per second, and let $\omega = 2\pi n$ be the free-oscillation angular velocity of the circuit, in radians per second. Then the reactance jX_l of the inductance DE at this frequency will be

$$jX_l = jl\omega \qquad \text{ohms} \quad (1)$$

where $j = \sqrt{-1}$, the quadrantal operator.

The reactance $-jX_c$ of the condenser at the same frequency will be

$$-jX_c = -j\frac{1}{c\omega} = -j\frac{s}{\omega} \qquad \text{ohms} \quad (2)$$

where $s = 1/c$ is the elastance of the condenser in darafs.[1]

Discharging and Charging Oscillations. — An oscillatory circuit may be excited into activity either when energy is added to it, or when energy is withdrawn from it. In the former case *charging oscillations*, and in the latter case *discharging oscillations* are produced. Thus the circuit may be initially devoid of electric or magnetic energy, and a certain constant potential difference of U_0 volts, as from a storage battery, may be inserted in the circuit between the terminals TT, Figure 1. The condenser will then be charged by an oscillatory process, or series of charging oscillations.

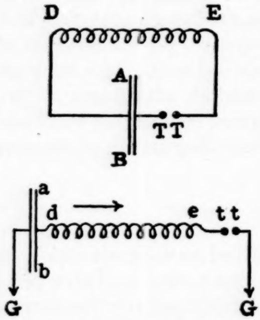


FIGURE 1. Diagram of connections of simple oscillating-current circuit, and schematic representation of the same.

Again, the condenser may be initially charged to a potential difference of U_0 volts, and allowed to discharge by short-circuiting the terminals TT. Discharging oscillations will then be produced. Or, the condenser may be initially without charge, but the inductance DE (Figure 1) may be charged by allowing a steady current of I_0 amperes to pass through it from some external source. If then the gap TT be closed, and the source of charging current is suddenly withdrawn sparklessly, discharging oscillations will be produced in the circuit.

Discharging oscillations may even be considered as taking place in an active initially energised circuit, in the limiting case where the circuit is assumed to be resistanceless; so that energy ceases to be dissipated during the discharge.

In any of the above cases, the circuit selects the frequency of free

[1] Prof. V. Karapetoff seems to have been the first writer to have had the courage to print the useful term *daraf* as a unit of elastance the reciprocal of a *farad*. "The Electric Circuit," Ithaca, N. Y. 1910, p. 72. See also "Electrical Papers," O. Heaviside, 1892, Vol. 2, p. 125.

oscillation such that the reactance of the circuit remains zero. This is the law of all oscillating-current circuits, whether they contain small or large resistance, up to the aperiodic limit. Consequently:

$$jX_l - jX_c = 0 \qquad \text{ohms} \quad (3)$$

$$jl\omega = j\frac{1}{c\omega} = j\frac{s}{\omega} \qquad \text{ohms} \quad (4)$$

whence

$$\omega = \frac{1}{\sqrt{lc}} = \sqrt{\frac{s}{l}}. \qquad \text{radians/sec.} \quad (5)$$

When condensers are connected in parallel, their permittances are more readily dealt with; and when connected in series, their elastances.

Discharging Oscillations of a Simple Resistanceless Oscillating-Current Circuit. — The oscillatory system of Figure 1 may be given its initial stock of energy either electrically or magnetically; that is, either by giving an initial electric charge of Q_0 coulombs to the condenser, or by exciting a total initial linked magnetic flux $\Phi_0 = I_0 l$ ampere-henrys in the coil, where I_0 is the initial exciting current-strength, supposed to be suddenly withdrawn from the coil without loss of energy in sparking. With respect to amplitude, the discharging oscillations of the coil will be the same as those of the condenser, provided that:

$$\Phi_0 = Q_0 z_0 \qquad \text{volt-seconds} \quad (6)$$

where $z_0 = \sqrt{\frac{l}{c}} = \sqrt{ls}$ is the surge impedance of the system. With respect to phase, however, the discharging oscillations of the excited coil will be in quadrature with those of the excited condenser. In cases where both the condenser and the coil are initially excited, and are allowed to discharge simultaneously, each may be considered independently, and the two sets of oscillations may then be summed.

General Rotating Vector-Diagram of Simple Resistanceless Oscillating-Current Discharging Circuit. — In Figure 2, let $O\overline{U}_0'$ represent to volt-scale the initial p. d. applied to the condenser of Figure 1, and producing therein an initial electric charge Q_0. Then $O\overline{U}_0$ will be the equal and opposite p. d. of U_0 volts, tending to discharge the condenser. The direction of the discharging p. d. OU_0 may be taken as the direction of reference or voltage phase-standard, and XOX as the axis of reference. The vector $O\overline{I}_0$ then represents the discharging current established by the discharging p. d. $O\overline{U}_0$. The vector system $O\overline{U}_0$, $O\overline{I}_0$, $O\overline{E}_0$ is to be pivoted about the point O, and, starting at time

$t = 0$ from the position shown, to rotate positively in the plane $\overline{U}_0\overline{I}_0\overline{E}_0$ with the angular velocity ω, as determined by formula (5). At any time t seconds after the release of the discharge, the orthogonal projections on XOX of the three vectors $O\overline{U}_0$, $O\overline{I}_0$, and $O\overline{E}_0$ will represent the corresponding instantaneous values of the discharging p. d. at condenser terminals, the current strength, and the emf. of self-induction in the coil. The vector discharging current therefore rotates in quadra-

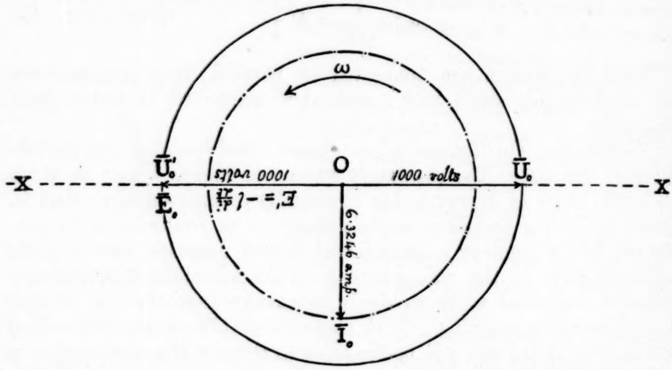


FIGURE 2. Rotative vector-diagram of simple resistanceless oscillating-current circuit.

ture between the two opposed and equal electromotive forces of discharge and of self-induction, developing with them reactive power and cyclic energy; but with no dissipated energy, under the assumption of negligible resistance.

Figure 3 presents a series of stationary vectors; ω, Y, I, P, W. The vector ω is drawn in the $-j$ direction, and with a length $\omega = 1/\sqrt{cl}$ $= \sqrt{s/l}$ radians-per-second, according to formula (5). If we take, as an example, a condenser of $c = 4$ microfarads ($s = 0.25$ megadarat), charged to an initial p. d. of $U_0 = 1000$ volts, and discharged through a resistanceless inductance of $l = 0.1$ henry, we find $\omega = 1581.14$ radians per second, corresponding to a frequency of $n = 251.646$ cycles per second, and a complete period of $T = 0.003,974$ second.

Starting from the initial position of Figure 2, the subsequent position of any vector in the system, after the lapse of t seconds, is determined by multiplying the vector by

$$\epsilon^{-(-j\omega t)} = \epsilon^{j\omega t} = \epsilon^{j\frac{t}{\sqrt{cl}}} = \epsilon^{j1581.14t}, \qquad \text{numeric} \angle \quad (7)$$

The admittance of the oscillating-current circuit is given by the Y vector, Figure 3, which is drawn in the $-j$ direction to a scale of mhos, and a length of $c\omega$ mhos. In the case considered $Y = -j0.006,324,6$ mho.

The initial vector oscillating current is given by the I vector, Figure 3, which is drawn in the $-j$ direction to a scale of amperes, and a length of $\overline{U}_0 Y$, as the amplitude, or maximum cyclic value, of the p. d. at condenser terminals. Or, expressed in terms of the quantity of electricity in the condenser, $\overline{I}_0 = \overline{Q}_0\omega$, where $\overline{Q}_0$ is the initial condenser charge in coulombs, and the vector amplitude or maximum cyclic quantity. $\overline{Q}_0$ is also $Q\sqrt{2}$, where Q is the root-mean-square of the oscillating condenser charge, in coulombs. In the case considered, $\overline{I}_0$ is $-j\ 6.3246$ amperes, and $I = 4.472,14$ r. m. s. amperes. This shows that in Figure 2 the current vector $O\overline{I}_0 = 6.3246$ max. cyclic amperes, lies 90° in phase behind the discharging p. d. OU_0.

The oscillatory power in the circuit is given by the P vector, Figure 3. Since the current is in quadrature with the emfs. in a resistanceless circuit, the power will be wholly reactive or non-dissipative. The P_m vector is drawn in the $-j$ direction, to a scale of watts, and to a

length of $P_m = \frac{\overline{U}_0 \overline{I}_0}{2} = UI$ units on this scale. In the case considered

$P_m = -j3162.3$ watts. This is the maximum cyclic value, or amplitude, of the power of the condenser. The power is positive when the condenser is doing work, or discharging, and is negative when the condenser is receiving energy from the magnetic field of the coil, or is charging.

The oscillatory energy in the circuit is given by the W vector. This vector is drawn in the $= -j$ direction, to a scale of joules, and to a length of $W_m = P/2\omega$ units on that scale. In the case considered, $W_m = -j1$ joule. This is the maximum cyclic value, or amplitude, of the energy delivered by the condenser into the magnetic field of the coil at each oscillation.

The time variation of the various oscillating quantities is represented in Figure 5, for one complete cycle of the current and p. d. The sinusoid u, u, u, of 1000 volts amplitude, is the graph of the condenser p. d., commencing at $U_0 = 1000$ volts. The sinusoid e is the emf. of self-induction in the coil, and is always equal and opposite to the p. d. of the condenser. The sinusoid ii, of 6.3246 amperes amplitude, is the graph of the discharging current, and is 90° in phase behind the discharging p.d. The sinusoid p, of double frequency, and of amplitude

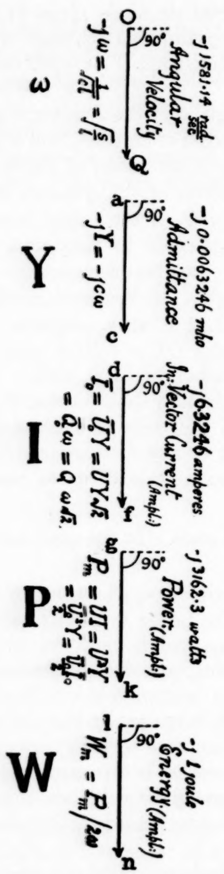


FIGURE 3. Stationary vector-diagrams of angular velocity, admittance, r. m. s., current strength, maximum cyclic power, and maximum cyclic energy in a simple resistanceless oscillating-current circuit to phase of p. d. as standard.

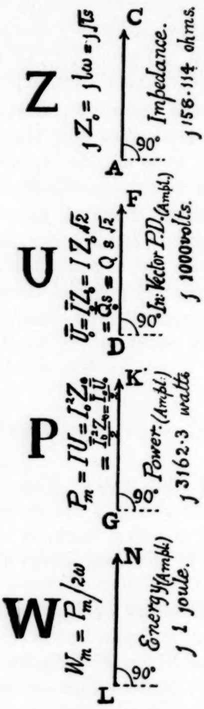


FIGURE 4. Stationary vector-diagrams of impedance, r. m. s., potential difference, maximum cyclic power, and maximum cyclic energy in a simple resistanceless oscillatory-current circuit to phase of current as standard.

3.1623 kilowatts, is the power developed in the condenser. It starts from zero, in the positive direction, at the moment of release. The opposite sinusoid p_l, of double frequency, and also of amplitude 3.1623 kw., is the power developed in the inductance. The condenser and coil are reciprocally and successively generator and recipient in respect to power.

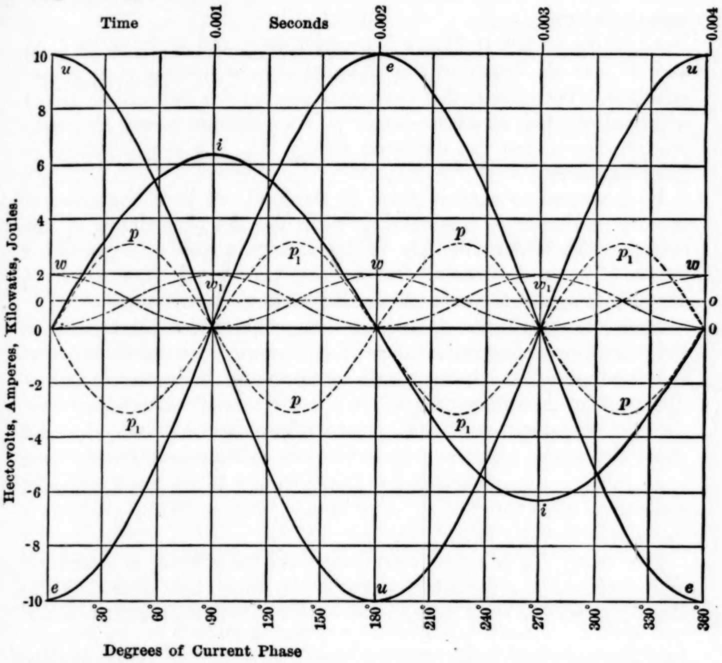


FIGURE 5. Analysis of p. d., current, power, and energy in a simple resistanceless discharging circuit.

The sinusoid w_l, of double frequency, is the graph of the energy expended by the condenser and stored in the coil. It has an amplitude of 1 joule above and below the line *oo* as axis. The total energy in the coil, therefore, reckoned from zero, is

$$2\ W_m = \frac{UI}{\omega} = \frac{\overline{U}_0 \overline{I}_0}{2\ \omega} = \frac{1}{2}\, l I_0^2. \qquad \text{joules} \quad (8)$$

The opposite sinusoid w_c, also of double frequency, and 1 joule amplitude, is the graph of energy in the condenser. It is evident that at every instant

$$w_c + w_l = W_0 \qquad \text{joules (9)}$$

where W_0 is the initial energy $U_0^2c/2$ of the system, or 2 joules in the case considered.

All of the vectors in Figure 3 are stationary, and in the series $\overline{I}$, P; and W, are all drawn to the phase of the discharging p. d. $\overline{U}_0$ as standard. This means that the vector current i is 90° behind that of u (Figure 5), the condenser power p is quadrature power, or purely reactive power, and the delivered energy W_m is also quadrature or reactive energy.

With respect to current-phase as standard, we have the series of vectors in Figure 4, commencing with Z_0 the impedance of the circuit, which is drawn in the $+j$ direction to a scale of ohms and a length of $l\omega$ on this scale. In the case considered, $Z_0 = j158.114$ ohms. Z_0 is thus a purely reactive impedance, or reactance.

The vector $\overline{U}_0$, Figure 4, represents the amplitude or initial value of the p. d. at condenser terminals. It is drawn in the $+j$ direction, or is 90° ahead of the current, and to a scale of volts, to a length of $\overline{I}_0 Z_0 = IZ_0\sqrt{2}$ on that scale, where I is the r.m. s. value of the vector $\overline{I}_0$. In the case considered, $\overline{I}_0 = 6.3246$ amplitude amperes and $I = 4.472$ r. m. s. amperes. In terms of the electric quantity, however, it may also be expressed as $\overline{Q}_0 s = Qs\sqrt{2}$ volts. In the case considered, this vector amplitude p. d. $\overline{U}_0$ is 1000 volts, with an effective or r. m. s. value of $U = 707.1$ volts.

The vector P_m in Figure 4 represents the amplitude, or maximum cyclic value, of the oscillatory power of the condenser. It is drawn in the $+j$ direction, being leading quadrature power with respect to current phase, to watt scale, and to a length of $\dfrac{\overline{I}_0 \overline{U}_0}{2} = IU$ units on that scale. In the case considered, $P_m = j\ 3162.3$ watts.

The vector W_m in Figure 4 represents the amplitude or maximum cyclic value of the oscillatory energy expended by the condenser. It differs only from the vector W_m of Figure 3, by being drawn in the $+j$ instead of in the $-j$ direction. This is because the energy and power are leading quadrature quantities with reference to the current, but lagging quadrature quantities with respect to the discharging p. d.

The stationary vector-diagrams of Figures 3 and 4 may be considered as graphically corresponding to the following vector equations:

With respect to the potential difference as standard of phase,

$$\bar{I}_0 = \bar{U}_0 Y \overline{|90^\circ} = \bar{Q}_0 \omega \overline{|90^\circ}, \qquad \text{maximum cyclic amperes}$$

$$P_m = \frac{\bar{U}_0 \bar{I}_0}{2} \overline{|90^\circ} = UI\overline{|90^\circ} = \frac{\bar{U}_0^2 Y}{2} \overline{|90^\circ} = U^2 Y \overline{|90^\circ}, \qquad \text{max. cyclic watts}$$

$$W_m = \frac{P_m}{2\,\omega} \overline{|90^\circ} = \frac{\bar{U}_0^2 Y}{4\,\omega} \overline{|90^\circ} = \frac{UI}{2\,\omega} \overline{|90^\circ}. \qquad \text{maximum cyclic joules}$$

With respect to the current as standard of phase :

$$\bar{U}_0 = \bar{I}_0 Z_0 \underline{|90^\circ} = \bar{Q}_0 s \underline{|90^\circ}, \qquad \text{maximum cyclic volts} \qquad (10)$$

$$P_m = \frac{\bar{I}_0 \bar{U}_0}{2} \underline{|90^\circ} = \frac{\bar{I}_0^2 Z_0}{2} \underline{|90^\circ} = IU \underline{|90^\circ} = I^2 Z_0 \underline{|90^\circ}, \qquad \text{max. cyc. watts} \qquad (11)$$

$$W_m = \frac{P_m}{2\,\omega} \underline{|90^\circ} = \frac{\bar{I}_0^2 Z_0}{4\omega} \underline{|90^\circ} = \frac{IU}{2\omega} \underline{|90^\circ}. \qquad \text{maximum cyclic joules} \qquad (12)$$

The vector-diagrams I, P, and W, of Figure 3, as well as U, P, and W, of Figure 4, may also be treated as rotative vector-diagrams, for deducing instantaneous values of their respective quantities projectively. Thus in the case of I, Figure 3., we may rotate the vector I_0 about the point d, in the positive direction, with angular velocity ω, commencing at time $t = 0$, from the position shown. The instantaneous orthogonal projections of the vector on the dotted line, or "real" axis, will then give the corresponding instantaneous o. c. strengths.

In the case of the P vector, Figure 3, the vector gk may be rotated about the point g with positively directed angular velocity $2\,\omega$, commencing from the position shown in Figure 3, at time $t = 0$. Instantaneous orthogonal projections on the dotted line, or "real" axis, will then give the corresponding instantaneous oscillating-current power, as indicated by the double-frequency sinusoid p in Figure 5. This is the power of the condenser. The same rotating power-vector may also be used to project the power of the inductance, if displaced in phase by 180°, that is, if it starts from rest in the diametrically opposite position to that shown in Figure 3.

In the case of the W vector, Figure 3, the vector ln may be rotated in the positive direction about the point l with angular velocity 2ω. If the vector starts, at time $t = 0$, from the position indicated in Figure 3, projections must be taken on the line ln, or axis of imaginaries. Instantaneous projections of the vector on this axis will then mark off instantaneous values of the energy in the condenser ($u^2c/2$ joules).

If the rotating vector be displaced 180° in phase, its projections will mark off instantaneous values of energy, ($i^2l/2$ joules) in the inductance. For reasons that will be evident on considering oscillating-current circuits containing resistance, it is preferable to describe a circle upon a diameter *LSC*, Figure 6, with the radius $ln = W_m$, Figure 3, and rotate this circle in the positive direction, with the angular velocity $2\,\omega$, at the same time rolling it along the axis OY, drawn through the point L. The orthogonal projection of the center S upon OY will mark time, as indicated in Figure 6 both to a scale of seconds, and to degrees of energy phase. The projection of the point C in the circle on the OX axis, commencing at c, will mark off a distance Oc corresponding to the instantaneous energy $u^2c/2$ joules in the condenser. Similarly, the projection of the opposite point L in the circle on the OX axis, commencing at l, will mark off a distance Ol corresponding, on the same scale, to the instantaneous energy $i^2l/2$ joules in the inductance.

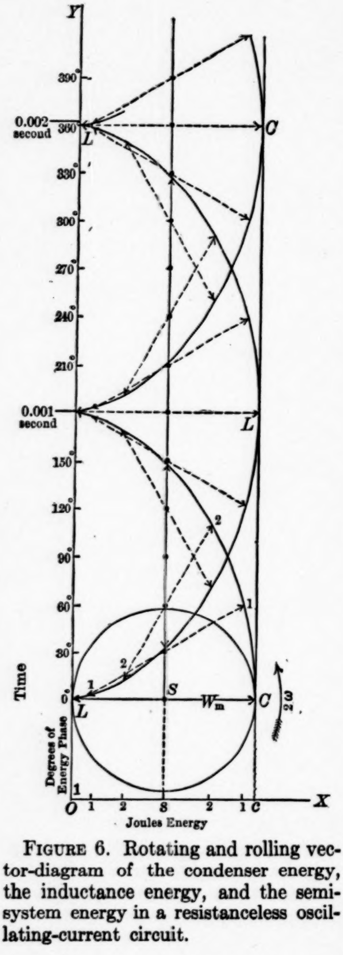


FIGURE 6. Rotating and rolling vector-diagram of the condenser energy, the inductance energy, and the semi-system energy in a resistanceless oscillating-current circuit.

If we connect the points L and C by a straight line, and take the middle point, it will coincide, in the resistanceless case here considered, with the center S of the rolling circle. Consequently, the projection of the point S on the OX axis at s, will mark off a length Os, equal to half the sum of the instantaneous energy in the condenser and in the inductance, i. e., equal to half the instantaneous

energy remaining in the system. We may call this quantity *Os*, for brevity, the semi-system-energy. Since there can be no dissipation of energy from such a circuit devoid of joulean and hertzian resistance, the semi-system-energy *Os* does not vary with time as the circle rolls along the OY axis. CCC and LLL are cycloids differing 180° in phase.

Inductance-Discharging Oscillations in a Simple Resistanceless Oscillating-Current Circuit. — We have thus far considered condenser-discharging oscillations. If, however, the inductance be charged with current and magnetic energy from a separate source, and this source is suddenly and sparklessly removed, while the condenser circuit Figure 1 is closed at $T\ T$, the inductance will set up a series of discharging oscillations. If we assume that the initial steady current strength I_0, at the moment of release, is equal to the maximum cyclic value of the current in the case already considered, then the oscillations of the inductance-discharging system will differ only in phase from the oscillations of the condenser-discharging system already discussed. Thus, if the current in the inductance were 6.3246 amperes at the instant of release, and the condenser were initially without charge, the oscillations of the system would be those of Figure 5, except that the time would start from the instant denoted by 90° in that diagram. If the initial direction of the current in the coil were reversed, the starting point in time would be at 270° in Figure 5. Consequently, with condenser-discharging oscillations, we start in Figure 5 with $t = 0$, either from the phase of 0° or of 180°, according to the direction of the p. d. impressed upon the condenser; while with inductance-discharging oscillations, we start with $t = 0$, either from the phase of 90° or of 270°, according to the direction of the current impressed upon the coil. The sequence of all the phenomena will then remain in each case as presented in Figure 5.

Not only Figures 2 and 5, but also Figures 3 and 4 apply equally to inductance and condenser-discharging oscillations. The diagrams of Figure 4 apply more directly, however, to inductance discharges, and those of Figure 3 to condenser discharges, because in the former the initial current is the independent common variable; while in the latter initial p. d. is the independent common variable. Thus, if in Figure 1 the inductance was initially charged with a current of 10 amperes, and an energy of 5 joules, the maximum cyclic current $\bar{I}_0$ would be 10 amperes, the r. m. s. or virtual current, I, 7.071 amperes. The maximum cyclic oscillating p. d. $\bar{U}_0$, in the absence of resistance, would be 1581.14 volts, the r. m. s., or virtual, p. d. U, would be 1118.0 volts, the maximum cyclic power P_m would be 7905.9 watts, and the maximum

cyclic condenser or inductance energy $7905.7/3162.3 = 2.5$ joules. The semi-system-energy would also be 2.5 joules.

Charging Oscillations in a Simple Resistanceless Oscillating-Current Circuit. — If with the system of Figure 1 initially unchanged, we suddenly impress upon the circuit, assumed resistanceless, between the terminals TT, a constant potential difference U_0, also assumed resistanceless, as from a storage battery of large cells, then both the condenser and the inductance will be subjected to charging oscillations. In this case, if the impressed p. d. U_0 is the same as that already assumed for the initial p. d. of the discharging condenser, the conditions represented in Figure 5 will apply to the charging oscillations, except in regard to phase, and to the meaning of the zero line OO. The sign of the oscillations will be reversed, or the phase displaced 180°, from those corresponding to a p. d. of the direction Ou, Figure 5. That is, an impressed p. d. having the + direction will set up from the start the same system of oscillations as those from the discharge of a condenser impressing a + direction of p. d. on the circuit. The zero line OO of Figure 5, in regard to p. d. and to condenser energy, instead of representing zero p. d. and zero condenser energy, will also have to be interpreted respectively in the charging case, as the constant value of impressed p. d., and the mean energy of the condenser under the impressed p. d. That is, the horizontal line through -10 will be the zero line of p. d. if the impressed p. d. is 1000 volts tending to make the condenser p. d. positive.

Simple Oscillating-Current Circuits Containing Resistance.

When such a circuit as that shown in Figure 1 is allowed to discharge through a known total resistance r ohms, including both joulean and hertzian resistances (all types that involve dissipation of power in proportion to the square of the current), the first step is to find the resistanceless angular velocity ω of Figure 3, that is to determine the angular velocity of discharge on the basis of no resistance ($r = o$). Let this value of resistanceless angular velocity be denoted by ω_0. We then proceed to determine the angular velocity ω in the presence of the actual resistance r.

Let $$\rho = r/2 \qquad \text{ohms} \quad (13)$$

be the semi-resistance of the circuit, and

$$\tau = l/\rho \qquad \text{seconds} \quad (14)$$

will then be a time-constant, which may, for convenience, be called the *oscillation time-constant* of the circuit, as distinguished from the ordi-

nary time-constant l/r when the circuit is non-oscillatory and the condenser is short-circuited. The oscillation time-constant is thus double the ordinary time-constant. If, as in Figure 7, we take $r = 200$ ohms in the same circuit as has been considered in Figures 3, 4 and 5, $\tau = 0.001$ second.

The time-constant reciprocal of the oscillating-current circuit is

$$\perp = \frac{1}{\tau} = \frac{\rho}{l} \cdot \text{ seconds}^{-1}. \quad (15)$$

In the ω-diagram of Figure 7, draw $OP = \perp$, to a suitable scale of reciprocal-seconds, in the direction of reference, or along the real axis in the positive direction. From P draw a line PQ in the $-j$ direction, or perpendicular to OP. With center O, and radius OQ equal, on the adopted scale, to the value of the resistanceless angular velocity

$$\omega_0 = \sqrt{s/l},$$

obtained as in Figure 3, intersect the line PQ in Q. Then the intercept PQ will measure, to scale, the angular velocity ω of the oscillation in the circuit with the resistance r present.

Or analytically,

$$\omega = \sqrt{\omega_0^2 - \perp^2} = \omega_0 \sin \phi \quad \text{radians per sec.} \quad (16)$$

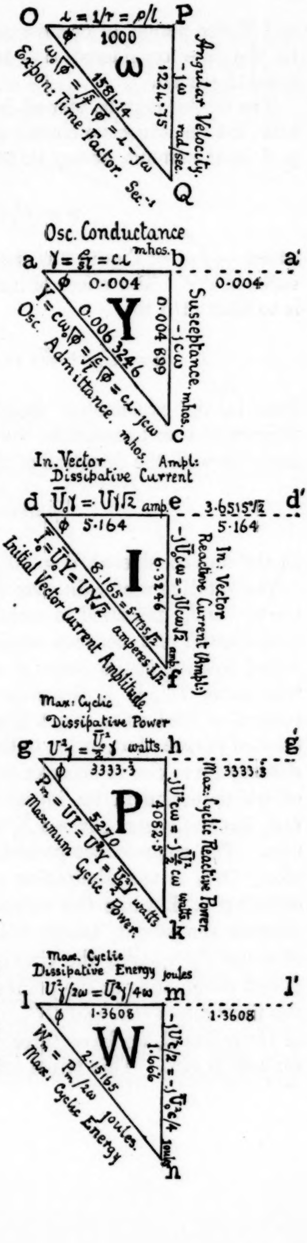


FIGURE 7. Stationary vector-diagrams of angular velocity, admittance, vector current strength, maximum cyclic power, and energy in a simple o. c. circuit containing resistance. Phase of p. d., standard.

and T, the period of current and p. d. is $2\pi/\omega$. seconds (17)
In the case considered $\omega = 1224.75$ radians per second, and $T = 0.00513$ second.

The matter may be viewed in another direction by considering that with the resistanceless circuit of Figure 3, the vector condition of the p. d. in the circuit, at any instant t seconds after release is

$$u = \bar{U}_0 \epsilon^{-t(-j\omega_0)} = \bar{U}_0 \epsilon^{j\omega_0 t} \qquad \text{volts} \quad (18)$$

where $-j\omega_0$ of Figure 3 is the factor of the time in the exponential variable $\epsilon^{j\omega_0 t}$. The effect of introducing a resistance r into the circuit is to alter (18) to

$$u = \bar{U}_0 \epsilon^{-t(\omega_0 \overline{|\phi})} = \bar{U}_0 \epsilon^{-t(\alpha - j\omega)} = \bar{U}_0 \epsilon^{-\alpha t + j\omega t}. \qquad \text{volts} \quad (19)$$

That is, the exponential time-factor $-j\omega_0$ of the resistanceless case (Figure 3) is deflected from the $-j$ direction to a direction making an angle $\overline{\backslash\phi}$ with the direction of reference; such that

$$\overline{\backslash\phi} = \cos^{-1}(\alpha/\omega_0). \qquad \text{radians or degrees} \quad (20)$$

In the case considered, $\phi = 50°.46'.06'' = 0.8861$ radian.

The rotative vector-diagram of the resistant circuit is shown in Figure 8. OU_0, measured to scale along the axis of projection $-XOX$, is the initial p. d. between condenser terminals at release. $O\bar{U}_0$ is the initial position of the vector p. d. whose projection is OU_0. In a certain sense, $O\bar{U}_0$ is a fictitious vector; because it has a value OU_0 cosec $\phi = 1291$ volts, which is greater than the initial p. d. at the moment of release; but, owing to the effect of damping, this seeming inconsistency gives rise to no error in the result. $O\bar{E}_0$ is the vector emf. of self-induction in its initial position. Midway between $O\bar{U}_0$ and $O\bar{E}_0$ lies the vector current $\bar{I}_0$, whose projection on XOX is initially zero. The cophase component Od (5.164 amperes), of the current along $O\bar{U}_0$, is the dissipative component taking power from the discharging p. d. while the component in quadrature thereto, $d\bar{I}_0$ is the reactive component, taking reactive power from $O\bar{U}_0$. The "drop" of vector $\bar{I}_0 r$ volts in the circuit would thus be a vector in line with $\bar{I}_0$ and terminating at the point r. This would also be the resultant of the two vectors $O\bar{E}_0$ and $O\bar{U}_0$. If we take a vector $-Ir$ drop $= OR$, or Or reversed, we have three vectors $O\bar{E}_0$, $O\bar{U}_0$, and OR whose vector sum is zero. This triple set of vectors is to be rotated about the

center O, with the uniform angular velocity ω, obtained from Figure 7. But instead of rotating in simple circles, as in the resistanceless case of Figure 2, the three vectors of Figure 8 rotate in equiangular spirals, the angle of each spiral being ϕ as defined in equation 20. That is,

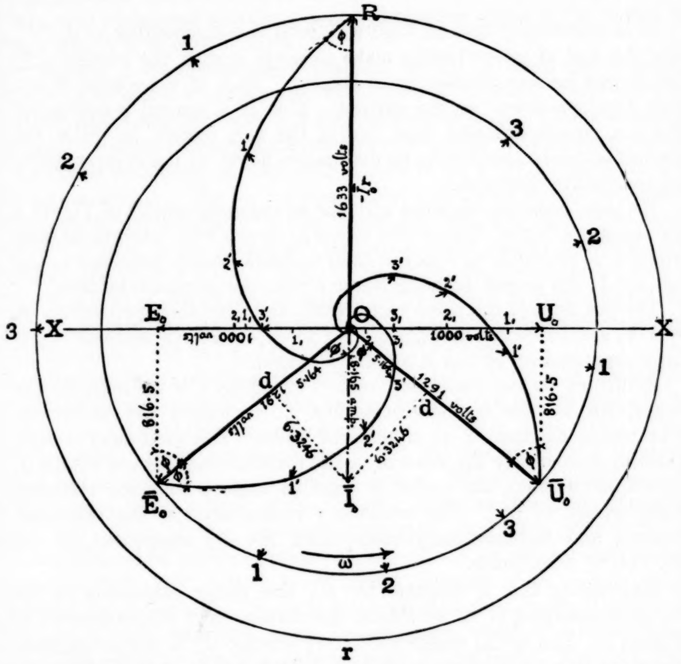


FIGURE 8. Rotative vector-diagram of an oscillating-current circuit containing resistance. Instant of release of condenser charge.

the tangent to the spiral at any point makes with the radius vector the constant angle ϕ. The vectors rotating with the uniform angular velocity ω of formula (16), describe equiangular spirals because energy is dissipated from the system in the resistance r, and each vector shrinks with time at the uniform exponential rate $\epsilon^{-\mu t}$; or falls to $1/\epsilon$th of its value in a time τ equal to the oscillatory time-constant. Since, however, all three vectors shrink at the same exponential rate, and since their vector sum in Figure 8 is initially zero, their vector sum will

always be zero, and the sums of their projections on the $-XX$ axis will also always be zero. That is, at all times

$$u - l\frac{di}{dt} - ir = 0. \qquad \text{volts} \quad (21)$$

It is noteworthy that in Figure 8, both the discharging p. d. $O\bar{U}_0$ and the emf. of self-induction make an angle ϕ with the current $O\bar{I}_0$, the former leading and the latter lagging. Each of these emfs. therefore develops power on the current. This is a general condition of the o. c. circuit, different from that of the a. c. circuit, in which the emf. of self-induction exerts no dissipative power on the current, being in quadrature therewith.

We may, however, dispense with the equiangular spirals of Figure 8 by assuming that all the vectors rotate in circles with uniform angular velocity ω, provided we apply to their instantaneously projected values on XOX, the proper damping factor $\epsilon^{-\alpha t}$ for the instant considered.

The positions of the three vector emfs. and also their projections on $-XX$, are indicated in Figure 8 for three successive instants angularly separated by 30°, or 0.000,427,5 second.

Returning to the stationary vectors of Figure 7, if we multiply the ω-diagram by the condenser-capacity c, we obtain the oscillatory admittance diagram Y to a scale of mhos. The oscillatory admittance is numerically the same as in the resistanceless case of Figure 3, (0.006,324,6 mho), but makes a negative angle ϕ with the reference axis, instead of 90°. The oscillatory conductance is the real component, and the oscillatory susceptance the $-j$ component, of the oscillatory admittance.

Multiplying the Y diagram by $\bar{U}_0$, the vector amplitude of the initial discharging p. d., we obtain the current or I diagram, $d\,e\,f$, of Figure 7. The initial vector oscillatory current $\bar{I}_0$ is 8.165 amperes, corresponding to a r. m. s. initial oscillatory current I of 5.7735 amperes. The reactive component $e\,f$ is the same as in the resistanceless case. The dissipative component $d\,e$ is the component in phase with the discharging p. d. $O\bar{U}_0$ Figure 8. A like component is also in phase with the self-inductive emf. OE; so that the total equivalent component of dissipation current would be $d\,d'$ Figure 7, or 10.328 amperes maximum cyclic initial value.

Multiplying the I diagram, Figure 7, by $\frac{\bar{U}_0}{2}$, we obtain the stationary-vector power-diagram P, or the watts triangle $g\,h\,k$, which may be drawn to any suitable scale of watts. This gives the maximum cyclic

power P_m, of which the component $h\ k$ is the maximum cyclic reactive power, or the maximum cyclic power in the inductance; while the real component $g\ h$ is the maximum cyclic power expended in resistance by the discharging p. d. $O\bar{U}_0$, Figure 8. But a like dissipation of power occurs under the influence of the emf. of self-induction $O\bar{E}_0$, so that the total undamped maximum cyclic dissipative power in the circuit is gg', Figure 7, of 6666.6 watts.

Finally, if we divide the P diagram by $2\,\omega$, or twice the resistant angular velocity, we obtain the W diagram of Figure 7, or the triangle $l\ m\ n$, which may be drawn to a suitable scale of joules. The stationary vector $l\ n$ is the undamped maximum cyclic energy W in the oscillatory circuit as measured, at condenser terminals, or 2.151,65 joules in the case considered. The $-j$ component, or $1.6\dot{6}$ joules, is the undamped maximum cyclic energy in the reactance, and the real component $l\ m = 1.3608$ is the undamped maximum cyclic energy dissipated by the discharging p. d. $O\bar{U}_0$, Figure 8, on the oscillatory current. But a like amount of energy will be dissipated by the self-inductive emf. OE. Consequently, the total undamped maximum cyclic dissipative energy in the circuit will be $l\ l'$, Figure 7, or 2.7216 joules.

The condition of either the vector p. d. $O\bar{U}_0$ (Figure 7), the vector self-inductive emf. $O\bar{E}_0$, or the vector current $O\bar{I}_0$ after t seconds, is obtained by applying the exponential $(\epsilon^{-(\Delta - j\omega)\,t})$, as in (19). This exponential may be expressed as $\epsilon^{-\Delta t}.\ \epsilon^{j\omega t}$, the first of which is a damping-factor, and the second a rotating factor. The diagrams of Figure 7 apply only to the effects of the rotating factor, as though no damping existed. That is, they represent undamped oscillating quantities, or quantities which would be projected on $-XOX$ by the rotation of the vectors $O\bar{U}_0$, $O\bar{E}_0$, OR, in pure circles, instead of in spirals. The damping-factor $\epsilon^{-\Delta t}$ to be applied, is represented in Figure 9, which is drawn on semi-logarithm paper, i.e., on paper ruled to a logarithmic scale of ordinates, but to a uniform scale of abscissas. The ordinates give the damping-factor, and the abscissas the time from release, to a scale of degrees, and also of seconds. The straight line OA represents the damping-factor for voltage and currents. The straight line OB represents the damping-factor for powers and energies. The time in which the voltage and current fall to $1/\epsilon$th of their initial value is τ, the oscillatory time-constant, or 0.001 second in the case considered. The number of radians through which the vectors of Figure 8 must rotate in order to shrink to $1/\epsilon$th of their initial values, i. e., the time-angle for a damping-factor of $\epsilon^{-1} = 0.367,88$, is $l\omega/\rho = \tan\phi$, or, in the case considered, 1.22475 radians $= 70.2°$.

Figure 10 shows the undamped time-values of the various quantities considered, or the values before applying the damping-factor. The sinusoid $\bar{u}$, of 1291 volts amplitude, corresponds, to the projection of

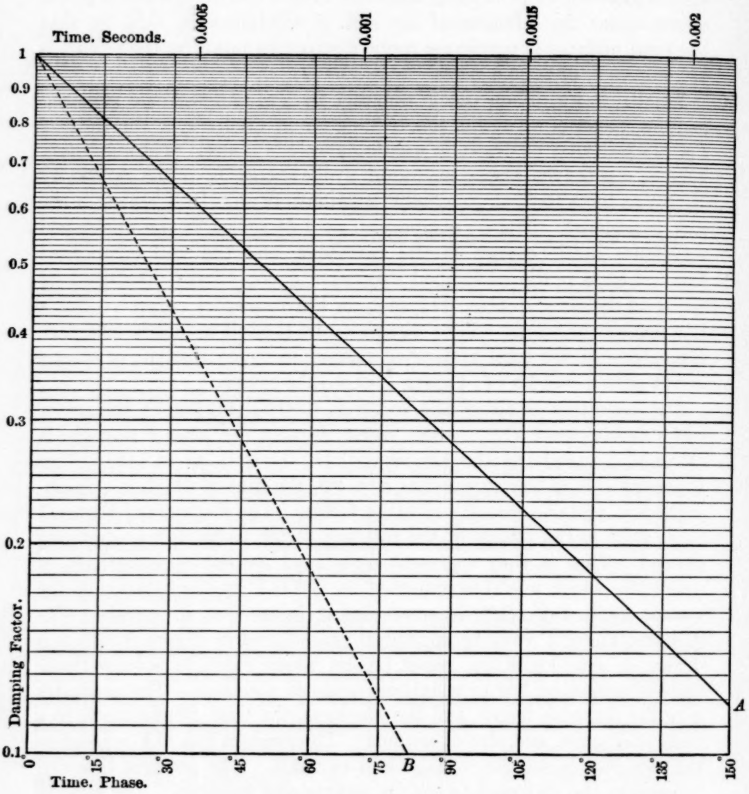


FIGURE 9. Damping-factors at different time-intervals after release.

$O\bar{U}_0$, Figure 8, on the XOX axis. Similarly, the sinusoid $\bar{e}$, of 1291 volts amplitude, corresponds to the projection of $O\bar{E}_0$, Figure 8, on the same axis. The heavy sinusoid $\bar{i}$, of 8.165 amperes amplitude, gives the undamped current, starting positively from zero at O. The emfs.

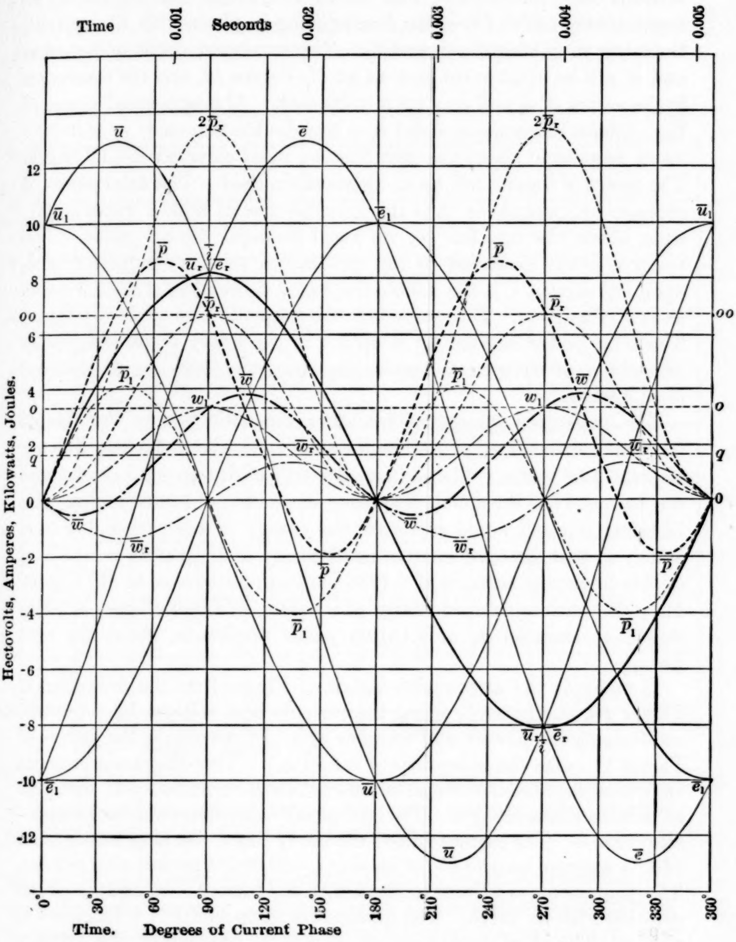


FIGURE 10. Curves of potential difference, current, power and energy in simple oscillating-current circuit containing resistance, leaving damping out of account.

$\bar{u}$ and $\bar{e}$, being out of phase with the current, are divided each into two components $\bar{u}_r$ $\bar{u}_l$ and $\bar{e}_r$ $\bar{e}_l$, the former being in phase with the current, the latter in quadrature therewith. The maximum cyclic values of $\bar{u}_r$ and $\bar{e}_r$ will be 816.5 volts each as at U, Figure 12, and the maximum cyclic values of $\bar{u}_l$ and $\bar{e}_l = 1000$ volts each. The undamped power of the cophase components $\bar{i}$ and $\bar{u}_r$ is the double-frequency $\bar{p}_r$ of 3333.3 watts amplitude about the zero line *oo*, itself elevated 3333.3 watts. The power of $\bar{i}$ and $\bar{e}_r$ will be an identical sinusoid. The total power of cophase components is thus the sinusoid 2 $\bar{p}_r$ of 6666.6 watts amplitude, about the zero line *oo*, *oo*, itself elevated 6666.6 watts. The reactive power expended by the quadrature voltage component of $\bar{u}_l$ upon the current $\bar{i}$, is the double-frequency sinusoid $\bar{p}_l$ of 4082.5 watts as at P, Figure 12. This power is in the magnetic field of the reactance. The total power exerted by $\bar{u}$ upon $\bar{i}$ is the heavy double-frequency sinusoid $\bar{p}$, of 5270 watts amplitude, about the zero line *oo* elevated 3333.3 watts.

The undamped energy of reactance magnetic flux is the double frequency sinusoid w_l, of $1.6\dot{6}$ joules amplitude, about the zero line *qq*, elevated $1.6\dot{6}$ joules. The undamped energy of dissipation in resistance due to $\bar{u}_r$ and $\bar{i}$, is the double-frequency sinusoid w_r of 1.3608 joules. An identical sinusoid would represent the energy of dissipation due to $\bar{e}_r$ and $\bar{i}$; so that the total unattenuated energy of dissipation would be a double-frequency sinusoid of 2.7216 joules amplitude, as at W, Figure 12. The total undamped energy of $\bar{u}$ acting on $\bar{i}$ is the heavy double-frequency sinusoid $\bar{w}$, of 2.151,65 joules amplitude, about the zero line *qq*.

If we apply the attenuation-factors of Figure 9 to the ordinates of Figure 10, that is, multiplying the currents and voltages by $^{-ut}$, while multiplying the powers and energies by ϵ^{-2ut}, we obtain the curves of Figure 11 (with the exceptions of w and w_r). This diagram represents the actual succession of events in the oscillating-current circuit. The p. d. at terminals falls along u. The emf. of self-induction pursues the opposite curve e. The current follows the heavy line i, reaching a maximum of 3.04 amperes near to 45° of its phase, or 0.00064 second after release. The components of voltage in phase with the current — u_r and e_r — both coincide with its curve. The quadrature components of voltage are u_l and e_l. The total component of voltage in phase with the current follows the curve marked 2 u_r. The power curve p reaches a maximum of 2200 watts at about 60° of its phase. This power rapidly subsides, and only crosses the zero line in feeble measure. The total dissipative power is shown by the curve 2 p_r.

The energy in the inductance follows the curve w_l. This corresponds to $\overline{w}_l$, in Figure 10, after applying the factor ϵ^{-2u}. The total energy dissipated in the resistance, and the total energy expended by the con-

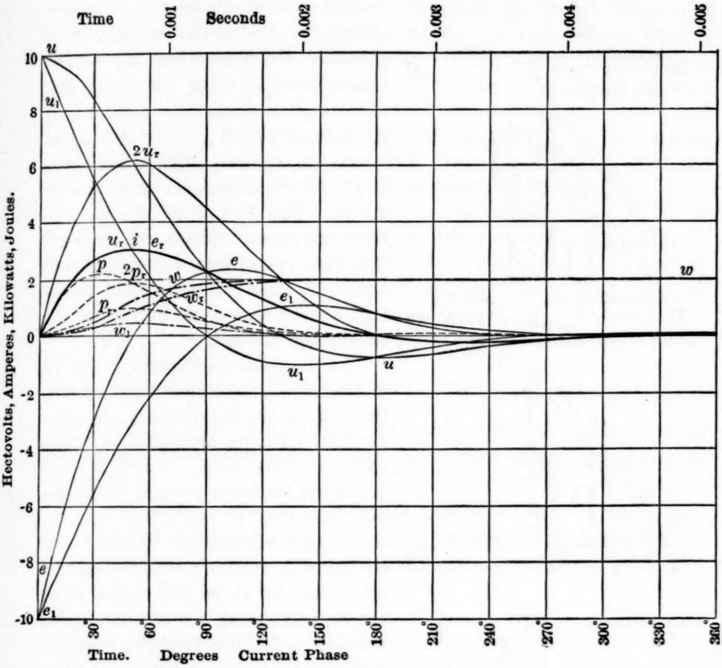
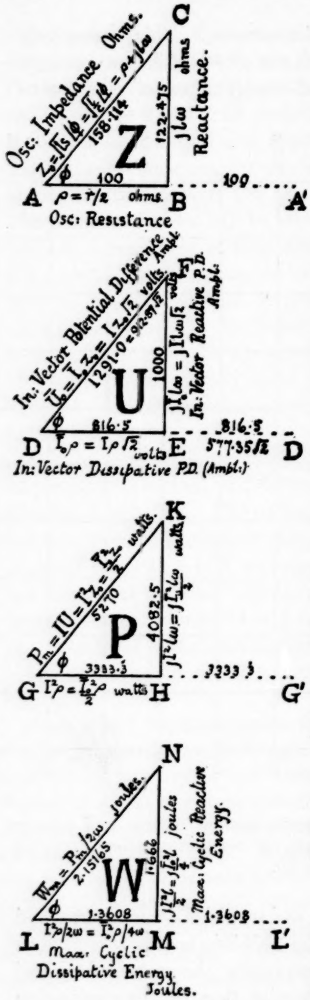


FIGURE 11. Curves of potential difference, current, power and energy in simple oscillating-current circuit containing resistance after applying damping-factors to the ordinates of Figure 10.

denser, follow respectively the curves w_r and w, which do not correspond with the cyclic curves $\overline{w}_r$ and $\overline{w}$ of Figure 10, being energy summations instead of instantaneous values. Consequently, the curves of $\overline{w}_r$, and $\overline{w}$ in Figure 10 must be interpreted otherwise than by the application of an attenuation-factor.

With respect to the phase of the oscillating-current as standard, we have the series of stationary vector-diagrams of Figure 12. The oscil-



latory impedance of the circuit is $Z = \rho + jl\omega$ ohms, or $Z_0 \underline{/\phi}$; where Z_0 is the impedance in the resistanceless condition of Figure 4. The total resistance r of the circuit is represented by AA'. The initial vector potential difference has an amplitude $\bar{U}_0 = \bar{I}_0 Z_0$ volts. The total initial drop of potential resistance is indicated by DD'. The undamped maximum cyclic power is indicated at P, and the undamped cyclic energy at W. The P and W diagrams to standard current phase are inverted with respect to those of standard p. d. phase, because the undamped power lags behind the p. d. but leads the undamped current.

The diagrams in Figures 7 and 12 of the oscillating-current circuit correspond to similar diagrams in connection with the alternating-current circuit.[2] The U, I, P, and W diagrams are stationary vector-diagrams, but they may be converted into rotating vector-diagrams. Thus I may be made a rotating vector-diagram by mounting it on the point d as center, and giving a positive angular velocity ω to the triangle, starting with df in the $-j$ direction. The projection of df on the real axis, after applying the damping-factor ϵ^{-ut}, will

FIGURE 12. Stationary vector-diagrams of impedance, vector potential difference, maximum cyclic power and energy in a simple o. c. circuit containing resistance. Phase of current, standard.

[2] "Vector Power in Alternating-Current Circuits" by A. E. Kennelly. Trans. American Inst. Elect. Engrs. June, 1910. Vol. 29.

give the actual current at any instant. This is equivalent to rotating df in an equiangular spiral of angle ϕ in the manner of Figure 8.

Similar treatment will convert U into a rotating vector-triangle, except that DF should start from the position $O\overline{U}_0$ of Figure 8.

The P diagram, in either Figure 7 or Figure 12, can be made into a rotating vector-diagram, by rotating the triangle with angular velocity 2ω about the vertex k or K respectively. Thus, taking the power triangle ghk of Figure 7, we mount it on an axis at k, Figure 13, and draw kr equal and parallel to hg. The three-vector system, kh, kg, and kr is then allowed to rotate with the angular velocity 2ω, in the case considered 2449.5 rad/sec, starting from the position shown, when the p. d. vector $O\overline{U}_0$, Figure 8, starts with angular velocity ω from the position KH, or parallel to OY. The orthogonal projections on $-XOX$ of the three vectors kh, kg, and kr, then define at any instant the undamped reactive, total, and dissipative power, under the action of $O\overline{U}_0$ on the oscillatory current. The total undamped power $\overline{p} = 5270$ watts thus lags $\frac{\pi}{2} + \phi$ behind the p. d. vector in terms of power phase, or $\frac{\pi}{4} + \frac{\phi}{2}$ in terms of p. d. phase (70° .23′). The total undamped power is measured on the OX axis from the point O and oscillates between the limits $O\overline{p} = 8603.3$ and $O{-}\overline{p} = -1936.6$ watts, as shown in Figure 10.

The undamped dissipative power $\overline{p}_r$ oscillates between the limits $O\overline{p}_r = 6666.6$ watts and zero. The total undamped dissipative power is $2\,\overline{p}_r = 13{,}333.3$ watts, owing to the separate actions of $O\overline{U}_0$ and $O\overline{E}_0$, Figure 8.

The undamped reactive power $\overline{p}_l$ is reckoned from o as zero, and oscillates between ± 4082.5 watts.

The damping-factor $\epsilon^{-2\mu t}$ must be applied to $\overline{p}$ and $\overline{p}_r$ as measured from O, and to $\overline{p}_l$ as measured from o, along the OX axis.

If we employ the phase of the oscillating current as standard, and as taken in Figure 10, we mount the power triangle GHK of Figure 12 on an axis at K, and rotate it, as before, at angular velocity 2ω, starting from the position GHK shown in Figure 13 when the current vector $O\overline{I}_0$ of Figure 8 is passing through the position KHo with angular velocity ω. The total power $\overline{p}$ will then lag $\frac{\pi}{2} - \phi$ behind the current vector, in terms of power-phase, or $\frac{\pi}{4} - \frac{\phi}{2}$, in terms of current phase

(19° 37′) as indicated in Figure 10. The $\bar{p}_r$ vector Kr then follows, 90° of power phase or 45° of current phase, behind the current at release. The vector KH, or undamped reactive power vector, Figure 13, starts in phase with the current.

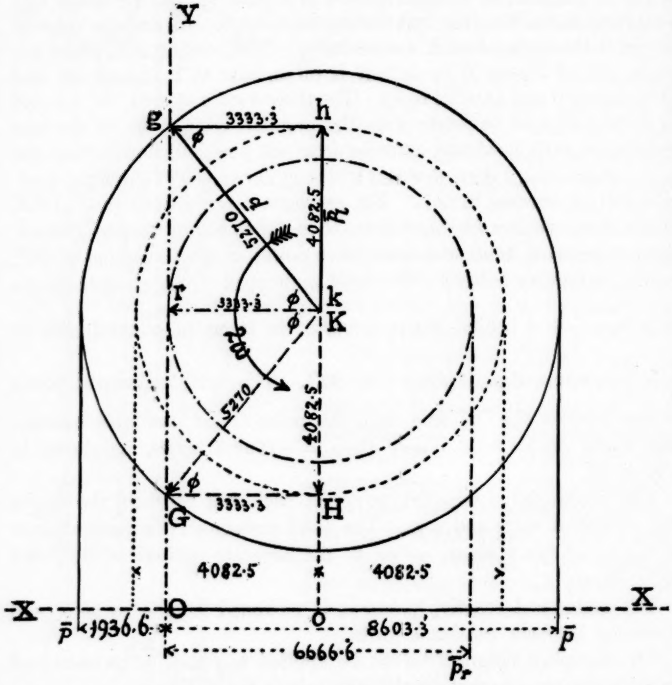


FIGURE 13. Rotative vector power diagram for p. d. standard phase and for current standard phase.

The stationary vector-diagrams of Figure 7 may be understood as involving the following phase relations, with respect to the phase of the undamped vector p. d. $O\bar{U}_0$, Figure 8.

$$\bar{I}_0 = \bar{U}_0 Y \,\overline{|\phi} = UY\sqrt{2}\,\overline{|\phi} = \bar{U}_0 c\omega_0 \,\overline{|\phi} \qquad \text{max. cyclic amperes} \quad (22)$$

$$P_m = UI\,\overline{|\phi} = \frac{\bar{U}_0 \bar{I}_0 Y}{2}\,\overline{|\phi} = \frac{\bar{U}_0^2}{2} Y\,\overline{|\phi} \qquad \text{max. cyclic watts} \quad (23)$$

That is, the undamped current lags ϕ° of p. d. phase behind the discharging undamped p. d. The undamped power reaches its positive

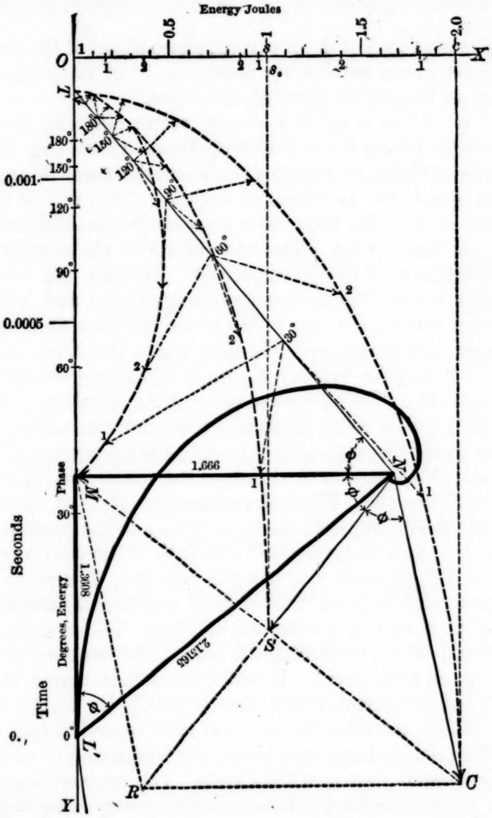


FIGURE 14. Rotating and rolling vector-diagram of the condenser energy, the inductance energy, the semi-system energy and the semi-dissipated energy in an oscillating-current circuit.

maximum ϕ° of its phase behind the discharging undamped p. d. Similarly in Figure 12, we may write the relations

$$\bar{U}_0 = \bar{I}_0 Z \,\underline{/\phi} = IZ\sqrt{2}\,\underline{/\phi} = \bar{I}_0 l\omega_0 \,\underline{/\phi} \qquad \text{max. cyclic volts} \quad (24)$$

$$P_m = IU \,\underline{/\phi} = \frac{\bar{I}_0^2 Z}{2}\,\underline{/\phi} = \frac{\bar{I}_0 \bar{U}_0 Z}{2}\,\underline{/\phi}. \qquad \text{max. cyclic watts} \quad (25)$$

Or the discharging p. d. is $\phi°$ of current phase ahead of the current and the undamped power reaches its maximum $\phi°$ of its phase ahead of the current, all being considered as undamped.

In order to derive a rotative energy vector-diagram, we take the triangle *lmn* of Figure 7, or $L'MN$ of Figure 12 and lay $L'M$ along the OY axis as shown in Figure 14. With the point N as pole, the equiangular spiral NL' is drawn, of angle ϕ. This will be tangent to the OY axis at L'. We then draw a vector $NC = NM$ and making with NM an angle of 2ϕ. The mid-point S of the straight line NC is then connected to N by the vector NS. We may call NC the condenser-energy vector, NM the reactive-energy vector, and NS the semi-system-energy vector; i. e., the vector of the half sum of the energy in the condenser and reactance. We now rotate the three vectors and the spiral, with angular velocity 2ω, while permitting the spiral to roll along the axis OY. The successive turns of the spiral are to be capable of rolling on this axis, as by displacing them infinitesimally out of the plane of the paper, like the wards of a conical band spring. The vectors NC, NS, and NM are also to shrink as they rotate by application of the damping-factor $\epsilon^{-2\iota t}$. Then the projections of C, M, and S, on the OX axis, will define the instantaneous energy in the condenser, reactance and semi-system respectively. The path of the pole N will be the straight line NT, pursued with damped velocity. The paths C and M will be exponential cycloids, that of S an exponential trochoid. The positions of C, S, and M are traced in Figure 14 for several energy phase intervals of 30°, or 0.000,213,8 second, the first two of which are marked 1, 2, on each curve. It will be seen that taking the energy scale along OX conformably with that of LMN, the condenser energy starts at 2 joules, and after 60° or 0.000,427,6 second, it falls to 1.3811 joules. The reactive energy *ol* commences at zero and after 60° power and energy phase it rises to 0.3544 joule. The semi-system energy *Os* starts at 1 joule, and after 60° falls to 0.8678 joule. The displacement $s_0 s$ is therefore half the dissipated energy = 0.1322 joule. The total dissipated energy at this instant is thus $2\, s_0 s = 0.2645$ joule. All three vectors finally terminate and shrink into the point T. The distance $TL' = NL'/\cos\phi$, and NT is perpendicular to NL'.

The fundamental differential equation for quantity q is satisfied by

$$q = A\epsilon^{-(\iota + j\omega)t} + B\epsilon^{-(\iota - j\omega)t}, \qquad \text{coulombs} \quad (26)$$

where A and B are integration constants, while ι and ω follow from the construction of the triangle OPQ, Figure 7.

Choosing the constants consistently with the discharge of the condenser initially charged to potential U_0 volts, the potential at time t is

$$u = U_0 \operatorname{cosec} \phi \, \epsilon^{-\iota t} \sin(\omega t + \phi) \quad \text{volts} \quad (27)$$

$$= \bar{U}_0 \epsilon^{-\iota t} \sin(\omega t + \phi), \quad \text{volts} \quad (28)$$

from which q follows by the relation $q = u/s = uc$ coulombs. $\bar{U}_0$ is the initial value of the vector discharging p. d. as obtained from Figures 8 and 12.

The instantaneous current is

$$i = Q_0 \omega \operatorname{cosec}^2 \phi \, \epsilon^{-\iota t} \sin \omega t \quad \text{amperes} \quad (29)$$

$$= U_0 c \omega_0 \operatorname{cosec} \phi \, \epsilon^{-\iota t} \sin \omega t \quad \text{amperes} \quad (30)$$

$$= \frac{\bar{U}_0}{z_0} \epsilon^{-\iota t} \sin \omega t = \bar{I}_0 \epsilon^{-\iota t} \sin \omega t, \quad \text{amperes} \quad (31)$$

where $$\bar{I}_0 = U_0 / l\omega = \bar{U}_0 / z_0 = \bar{U}_0 c \omega_0. \quad \text{amperes} \quad (32)$$

The emf. of self-induction in the circuit at any instant is

$$e = U_0 \operatorname{cosec} \phi \, \epsilon^{-\iota t} \sin(\omega t - \phi) \quad \text{volts} \quad (33)$$

$$= \bar{U}_0 \epsilon^{-\iota t} \sin(\omega t - \phi). \quad \text{volts} \quad (34)$$

The instantaneous power of the condenser in the circuit is

$$p = ui = \bar{U}_0 \bar{I}_0 \epsilon^{-2\iota t} \sin \omega t \cdot \sin(\omega t + \phi) \quad \text{watts} \quad (35)$$

$$= \frac{\bar{U}_0 \bar{I}_0}{2} \epsilon^{-2\iota t} [\cos \phi - \cos(2\,\omega t + \phi)] \quad \text{watts} \quad (36)$$

$$= UI \epsilon^{-2\iota t} [\cos \phi - \cos(2\,\omega t + \phi)]. \quad \text{watts} \quad (37)$$

The instantaneous power of the reactance in the circuit is

$$p' = ei = \bar{U}_0 \bar{I}_0 \epsilon^{-2\iota t} \sin \omega t \cdot \sin(\omega t - \phi) \quad \text{watts} \quad (38)$$

$$= \frac{\bar{U}_0 \bar{I}_0}{2} \epsilon^{-2\iota t} [\cos \phi - \cos(2\,\omega t - \phi)] \quad \text{watts} \quad (39)$$

$$= UI \epsilon^{-2\iota t} [\cos \phi - \cos(2\,\omega t - \phi)]. \quad \text{watts} \quad (40)$$

The total dissipation power of the condenser and reactance is after t seconds,

$$2p_r = p + p' = \bar{U}_0^2 \gamma \epsilon^{-2\mu t} (1 - \cos 2\omega t), \qquad \text{watts} \quad (41)$$

$$= \bar{I}_0^2 \rho \epsilon^{-2\mu t} (1 - \cos 2\,\omega t), \qquad \text{watts} \quad (41a)$$

where $\gamma = c\mu$, the oscillation conductance of the circuit. The energy in the condenser t seconds after release, if $W_l = mn$, Figure 7, $= MN$, Figure 12, is

$$w_c = \frac{\bar{U}_0^2}{4} c \epsilon^{-2\mu t} [1 - \cos (2\,\omega t + 2\,\phi)] \qquad \text{joules} \quad (42)$$

$$= W_l \epsilon^{-2\mu t} [1 - \cos (2\,\omega t + 2\phi)]. \qquad \text{joules} \quad (43)$$

The energy in the reactance t seconds after release is

$$w_l = \frac{\bar{U}_0^2}{4} c \epsilon^{-2\mu t} (1 - \cos 2\,\omega t) \qquad \text{joules} \quad (44)$$

$$= W_l \epsilon^{-2\mu t} (1 - \cos 2\,\omega t). \qquad \text{joules} \quad (45)$$

The total energy of the system t seconds after release is

$$w = w_c + w_l = \frac{\bar{U}_0^2}{2} c \epsilon^{-2\mu t} [1 - \cos (2\,\omega t + \phi) \cos \phi] \quad \text{joules} \quad (46)$$

$$= 2\,W_l \epsilon^{-2\mu t} [1 - \cos (2\,\omega t + \phi) \cos \phi]. \quad \text{joules} \quad (47)$$

The semi-system energy w_s at time t is therefore

$$w_s = W_l \epsilon^{-2\mu t} [1 - \cos (2\,\omega t + \phi) \cos \phi]. \quad \text{joules} \quad (48)$$

The initial energy of the system is

$$W_0 = \frac{\bar{U}_0^2 c}{2} \sin^2 \phi = 2\,W_l \sin^2 \phi = 2\,W_m \sin^2 \phi = \frac{U_0^2 c}{2}. \quad \text{joules} \quad (49)$$

The total loss by dissipation is at any instant

$$w_d = W_0 - w$$

$$= \frac{\bar{U}_0^2 c}{2} \{\sin^2 \phi - [1 - \cos (2\,\omega t + \phi) \cos \phi] \epsilon^{-2\mu t}\}. \quad \text{joules} \quad (50)$$

This is the total expenditure of energy in I^2r up to time t. At any complete energy cycle, when $\cos(2\omega t + \phi) = \cos\phi$, and $t = m\frac{T}{2}$,

$$w_d = \frac{\bar{U}_0^2 c}{2}\left[\sin^2\phi\left(1 - \epsilon^{-2\iota m\frac{T}{2}}\right)\right] = W_0\left(1 - \epsilon^{-\iota mT}\right). \quad \text{joules} \quad (51)$$

The energy dissipated in the first energy cycle, when $t = T/2$,

$$w_1 = W_0\left(1 - \epsilon^{-\iota T}\right). \quad \text{joules} \quad (52)$$

The dissipation in successive energy cycles 1st, 2d, 3rd, 4th, etc., is

$$w_1, \quad w_1\epsilon^{-\iota T}, \quad w_1\epsilon^{-2\iota T}, \quad w_1\epsilon^{-3\iota T}, \quad \text{etc.} \quad \text{joules} \quad (53)$$

The total ultimate energy dissipated is

$$W_0 = w_1\left(1 + \epsilon^{-\iota T} + \epsilon^{-2\iota T} + \ldots\right) \quad \text{joules} \quad (54)$$

$$= \frac{w_1}{1 - \epsilon^{-\iota T}} \quad \text{joules} \quad (55)$$

With the values in the case considered of $\bar{U}_0 = 1291$, $U_0 = 1000$, $c = 4 \times 10^{-6}$, $\omega = 1224.75$, $\phi = 50°\ 46'$, $\iota = 1000$, $T = 0.00513$, $W_0 = 2$, we have, for the attenuation factor of one power period or semi-period of p. d., $\epsilon^{-\iota T} = 0.005{,}987$. The energy dissipated by I^2r in the first energy cycle is thus $2 \times 0.994{,}01 = 1.988{,}02$ joules. The second cycle dissipates $0.005{,}987 \times 1.988{,}02 = 0.0119$ joule. Each successive cycle dissipates 0.5987 per cent of the amount dissipated in the last preceding cycle. It is thus evident that in a damped oscillatory discharge, a relatively large fraction of the energy is rejected from the system in the first half-cycle of voltage or current, i. e., the first complete energy cycle.

Logarithmic Decrement.

If v be any vector oscillating-quantity of the type

$$v = V_0\epsilon^{-\iota t}\sin(\omega t + \phi), \quad \text{Ph. Q.} \quad (56)$$

such as a voltage, current, or force.

Then the rotating vector of this quantity $V_0\epsilon^{-\iota t}\epsilon^{j\omega t}$, in passing from one assigned position to another, in a time t_1 seconds, decreases from the first to the second value by the exponential, or damping factor, $\epsilon^{-\iota t_1}$. The exponent, ιt_1, of the damping may be defined as the Nape-

rian logarithm of the decrement during the interval, or simply as the log-decrement in the interval. If the log-decrement be denoted by δ, then

$$\delta = \perp t_1. \qquad \text{numeric} \quad (57)$$

If the rotating vector moves through one radian at the actual angular velocity ω, the time t_1 occupied in the passage will be $1/\omega$ seconds; so that

$$\delta_1 = \frac{\perp}{\omega} = \cot \phi. \qquad \text{numeric} \quad (58)$$

If the rotating vector moves through a half-cycle, semi-revolution, or π radians, the time occupied in the passage will be $t_1 = \pi/\omega$ seconds; so that

$$\delta_\pi = \pi \frac{\perp}{\omega} = \frac{\perp}{2\,n} = \pi \cot \phi, \qquad {}^{3}\,\text{numeric} \quad (59)$$

which is the log-decrement of any two successive elongations of the vector's projection in opposite directions on the axis of reference XX. If the vector moves through a whole cycle, revolution, or $2\,\pi$ radians, the time occupied will be $t_1 = 2\,\pi/\omega$ seconds, and

$$\delta_{2\pi} = 2\,\pi \frac{\perp}{\omega} = \frac{\perp}{n} = \frac{T}{\tau} = 2\,\pi \cot \phi, \qquad \text{numeric} \quad (60)$$

which is the log-decrement of any two successive elongations of the vector's projection in one and the same direction on the reference axis. Consequently, from any pair of successive maxima of the oscillating quantity v, the log-decrement is obtainable, and from this the angle ϕ of the equiangular spiral and of the stationary vectors for the quantity is obtainable. In the case considered, $\delta_1 = 0.8165$, $\delta_\pi = 2.5651$, and $\delta_{2\pi} = 5.1302$.

Root of Mean Square of Oscillating-Current Quantities.

If v be any vector oscillating quantity of the type $V_0 \epsilon^{-\perp t} \sin \omega t$, such as an oscillating-voltage, current, or force; then if the damping coefficient $\perp$ is taken as zero, the square root of the mean square of the quantity during any integral number of cycles is $V = V_0/\sqrt{2}$, as in ordinary alternating-current theory. If, for instance, the initial voltage is 1000, and there were no damping, the r. m. s. voltage of the o. c.

[3] The condition expressed in (59) was first pointed out by Prof. Clerk-Maxwell, in a somewhat different manner. See Bibliography.

circuit as shown by an ideally perfect voltmeter would be 707.1 volts. The question is what would such an instrument show in the presence of a known damping? The reading of the instrument would evidently fall as the damping continued, and, in most practical cases, would fall very rapidly; so that the inquiry must be limited to a certain definite instant during the charge or discharge; or at least to a certain definite interval of time within the process. At any instant we have

$$v^2 = V_0^2 \epsilon^{-2\perp t} \sin^2 \omega t. \qquad \text{(Ph. Q.)}^2 \quad (61)$$

The time integral of this square from time $t = 0$ to $t = \infty$ is

$$\int_0^\infty v^2 dt = \frac{V_0^2}{4\perp}\left(\frac{\omega^2}{\perp^2 + \omega^2}\right) = \frac{V_0^2}{4\perp}(1 - \cos^2 \phi). \qquad \text{(Ph. Q.)}^2 \text{ secs.} \quad (62)$$

But after the lapse of a time t_1 seconds, including m complete periods of the oscillating quantity, its value will have become

$$v_{mT} = V_0 \epsilon^{-\perp m T} \sin \omega t = V_0 \epsilon^{-\perp t_1} \sin \omega t, \qquad \text{Ph. Q.} \quad (63)$$

where T is the period of the oscillation, and m is any positive integer; so that

$$\int_0^{t_1} v^2 dt = \frac{V_0^2}{4\perp}(1 - \cos^2 \phi)(1 - \epsilon^{-2\perp t_1}). \qquad \text{(Ph. Q.)}^2 \text{ secs.} \quad (64)$$

The mean of this square during the interval is

$$\frac{1}{t_1}\int_0^{t_1} v^2 dt = \frac{V_0^2}{4\perp t_1}(1 - \cos^2 \phi)(1 - \epsilon^{-2\perp t_1}), \qquad \text{(Ph. Q)}^2 \quad (65)$$

and the root of this mean square during the interval is

$$V = \sqrt{\frac{1}{t_1}\int_0^{t_1} v^2 dt} = \frac{V_0}{\sqrt{2}}\left(\sin \phi \sqrt{\frac{1 - \epsilon^{-2\perp t_1}}{2\perp t_1}}\right)$$

$$= \frac{V_0 \epsilon^{-\perp \frac{t_1}{2}}}{\sqrt{2}}\left(\sin \phi \sqrt{\frac{\sinh \perp t_1}{\perp t_1}}\right). \qquad \text{Ph. Q.} \quad (66)$$

But $V_0/\sqrt{2}$ is the r. m. s. value of the undamped oscillating quantity, and $V_0 \epsilon^{-\perp \frac{t_1}{2}}/\sqrt{2}$ would be the r. m. s. value of the oscillating quantity at the mid-interval if it were to continue thereat throughout un-

damped. Consequently the r. m. s. value of an oscillating quantity $V_0\epsilon^{-\alpha t}\sin\omega t$ during any number of complete periods, is either the r. m. s. value of its initial undamped vector multiplied by

$$\left(\sin\phi\sqrt{\frac{1-\epsilon^{-2\alpha t_1}}{2\alpha t_1}}\right)$$

or the r. m. s. value of its mid-interval damped vector, multiplied by $\sin\phi\sqrt{\frac{\sinh\alpha t_1}{\alpha t_1}}$. After αt_1 passes the numeric 2, the first of these rapidly converges to $\frac{\sin\phi}{\sqrt{2\alpha t_1}}$, so that when the oscillatory damping is rapid, the r. m. s. value of the oscillatory quantity varies inversely as the square root of the interval during which the summation is effected. If the summation be confined to a single period, starting with radius vector V_0, $m = 1$ and

$$V = \sqrt{\frac{1}{T}\int_0^T v^2 dt} = \frac{V_0}{\sqrt{2}}\left(\sin\phi\sqrt{\frac{1-\epsilon^{-2\alpha T}}{2\alpha T}}\right)$$

$$= \frac{V_0\epsilon^{-\alpha\frac{T}{2}}}{\sqrt{2}}\left(\sin\phi\sqrt{\frac{\sinh\alpha T}{\alpha T}}\right). \qquad \text{Ph. Q.} \quad (67)$$

In the case considered, the current is $i = 8.165\ 2^{-1000t}\sin 1224.75t$. If we take the first complete period of $T = 0.00513$ second, with

$$\sin\phi = 0.7746;\ \text{and}\left(\sin\phi\sqrt{\frac{1-\epsilon^{-2\alpha T}}{2\alpha T}}\right) = 0.7746\sqrt{\frac{1}{10.26}} = 0.242;$$

so that the r. m. s. value of the current during the first complete damped oscillation would be 1.396 amperes.

If the oscillating quantity is a cosinusoid of the type

$$v = V_0\epsilon^{-\alpha t}\cos\omega t. \qquad \text{Ph. Q.} \quad (68)$$

The integral to infinite time of its square:

$$\int_0^\infty v^2 dt = \frac{V_0^2}{4\alpha}(1+\cos^2\phi). \qquad (\text{Ph. Q.})^2 \quad (69)$$

So that by the same reasoning as before

$$V = \sqrt{\frac{1}{t_1}\int_0^{t_1} v^2 dt} = \frac{V_0}{\sqrt{2}}\sqrt{\frac{(1 + \cos^2\phi)(1 - \epsilon^{-2\alpha t_1})}{2\alpha t_1}}$$

$$= \frac{V_0}{\sqrt{2}}\epsilon^{-\alpha\frac{t_1}{2}}\sqrt{(1 + \cos^2\phi)\left(\frac{\sinh \alpha t_1}{\alpha t_1}\right)}, \quad \text{Ph. Q.} \quad (70)$$

which differs from the preceding corresponding expression (66) in the substitution of $\sqrt{1 + \cos^2\phi}$ for $\sqrt{1 - \cos^2\phi}$. By the combination of

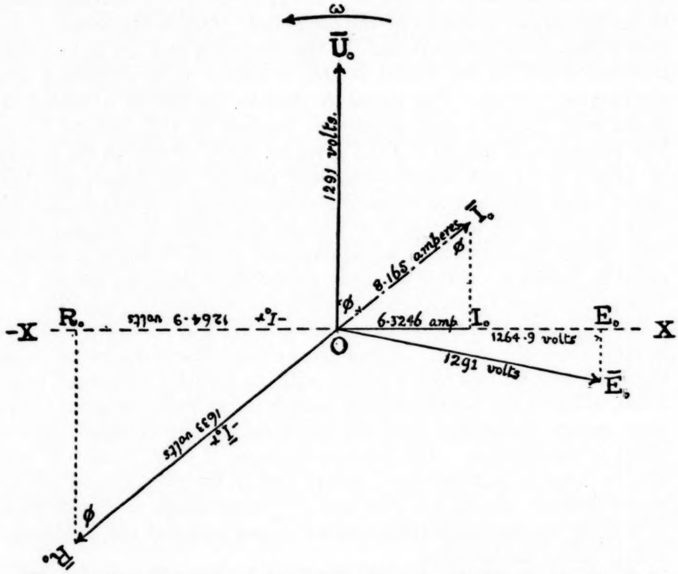


FIGURE 15. Rotative vector-diagram of an oscillating-current circuit containing resistance. Instant of release of reactance charge.

formulas (66) and (70) the r. m. s. values of oscillating quantities containing a sum of sine and cosine terms can be evaluated.

Discharging Oscillations from Reactance. — If with the condenser (Figure 1) uncharged, and resistance r in the circuit, we charge the reactance with magnetic energy, by passing through it a steady current of I_0 amperes, we may then allow the reactance to discharge through the condenser and resistance. We first find ω_0 the angular velocity of

discharge for $r = 0$ and then determine ω by the stationary vector-diagram of Figure 7, taking the full resistance r into account.

Upon the axis $-XOX$ of reference, Figure 15, lay off the initial value OI_0 of the current in the reactance, at the instant of release, assumed in this case to be 6.3246 amperes. The initial energy in the reactance will be $lI_0^2/2 = 2$ joules. Lay off an angle $XO\bar{I}_0 = 90_0 - \phi$, and a vector initial current $\bar{I}_0 = I_0$ cosec $\phi = 8.165$ amperes. All of the diagrams in Figures 7 and 12 now apply. The U diagram gives us the initial position of the condenser p. d. vector $O\bar{U}_0, = 1291$ volts. The projection of this on $-XOX$ is $OU_0 = 0$, corresponding to the uncharged condition of the condenser. $O\bar{E}_0$ is the initial position of the vector inductive emf. in the reactance, and the initial self-inductance emf. in the circuit is $OE_0 = 1264.9$ volts propelling the discharging current. The initial Ir drop in the circuit coincides in phase with $\bar{I}_0$, and taken negatively, extends to $O\bar{R}_0 = 1633$ volts. The projection of this on $-XOX$ gives an initial Ir emf. in the circuit of 1264.9 volts just equilibrating the emf. of self-induction. The entire vector system is to be considered as starting to rotate at angular velocity ω. If the diagram is to include the effects of damping, then each vector must rotate in an equiangular spiral of angle ϕ as indicated in Figure 8. But if we apply independent damping factors, the vectors in Figure 15 may rotate in circles as undamped quantities.

It will be seen that Figure 15 corresponds to Figure 8 except that it is advanced $\pi - \phi$ in phase. Thus a given energy charge discharged from the condenser, in this case 2 joules, with an initially uncharged reactance, will give rise to precisely the same rotative vector-diagram as the same energy discharged from the reactance, except in regard to the phase of the diagram. The rotative diagrams of U, I, P, and W will also be the same in either case; except that in the rotating and rolling vector-diagram Figure 14, NM and NC interchange in significance, NM being the reactance-energy vector in one case and the condenser-energy vector in the other.

Simultaneous Discharging Oscillations from Condenser and Reactance. — It is possible for a circuit like that of Figure 1, containing resistance, inductance, and capacity in simple series, to be released with an electric charge in the condenser, and an independent magnetic charge in the coil. The discharge which follows is then in part a condenser discharge, and in part a reactance discharge. If we know the initial charges, we can obtain the vector-diagram of the discharge by determining the ω-diagram of the system (Figure 7) and then making two separate rotative vector-diagrams, one, like Figure 8, for the dis-

charging condenser and the other, like Figure 15, for the discharging reactance. These two vector-diagrams are now to be combined vectorially, into a new resultant vector-diagram, which will represent the behavior of the mixed discharge. Since each of the component diagrams obeys the geometry of Figures 7 and 12, the resultant diagram will also obey it. The rotative vector-diagrams U, I, P, and W will also follow, but the W diagram of Figure 13 will be ambiguous, except in regard to the dissipated energy.

It is evident, moreover, that since the energy of a simple resistance-reactance-condensance oscillator exchanges harmonically from the electric to the magnetic form, after correcting for dissipation, any initial state of assigned separate electric and magnetic energies must correspond to some phase of a discharge from a certain single stock of energy, electric or magnetic.

Charging Oscillations of Circuit Containing Resistance. — If in the circuit of Figure 1, assumed to possess resistance, and with no initial charge, we insert a constant charging p. d. between the terminals t, t, in such a direction as will cause a subsequent discharge to flow in the positive sense of the arrow $d\ e$; then both the charging emf. and the initial direction of the charging current must be reckoned negative, or in the sense e, d. The charging vector-diagram will then be the same as that of Figure 8, with a phase difference of 180^0, or read upside down. The stationary vector-diagrams of Figures 7 and 12 apply as before, as well as the rotative vector-diagrams of Figures 13 and 14. In the last-named, however, the condenser energy Oc must be counted from c to O, or in the reversed direction, since the energy in the condenser is initially nil and increases to its final full value.

We may consider that Figure 8 is the vector-diagram of a charging condenser system in which the constant impressed emf. between the terminals $t\ t$ (Figure 1) is positive, or in the direction of the arrow $d\ e$. If as before $c = 4 \times 10^{-6}$ farad; $r = 200$ ohms; $l = 0.1$ henry and $E = 1000$ volts, we first find $\omega_0 = 1581.14$ radians per second as in Figure 3, then the angle ϕ, and $\omega = 1581.14 \ \overline{\backslash 50^\circ\ 46'}$, as in Figure 7.

We then lay off $OU_0 = E = 1000$ volts in Figure 8 and $O\bar{U}_0 = OU$.

$\operatorname{cosec} \phi \ \overline{\backslash \frac{\pi}{2} - \phi} = 1291 \ \overline{\backslash 39^\circ\ 14'}$ volts. This is the vector charging p. d.

The vector current follows from Figure 7 as 8.165 amperes, and the vector emf. of self-induction symmetrical with OU_0 in regard to the current. The graphs of Figures 10 and 11 then apply as before, except that in Figure 11 the p. d. must be read from the top line as zero, since the potential of the condenser at d, Figure 1, commences at zero and

falls to a final value of — 1000 volts. The other graphs in Figure 11 remain unaltered.

The charging oscillations of a simple circuit containing resistance differ only from the discharging oscillations with the same resistance, in regard to phase direction, and to the absolute numerical values of the condenser potential u, condenser charge q, and condenser energy W_0.

Condensers and Reactances in Series.

If we have a circuit (Figure 16), containing a plurality of condensers, or of reactances, or of both, in simple series, containing also resistances, and subject to charging or discharging oscillations, we may compute the behavior of the system as follows :

Let r' = sum of the individual joulean resistances. (ohms).
r'' = sum of the individual hertzian resistances. (ohms).
$r = r' + r''$ = the sum of all the individual resistances. (ohms).
$s = s_1 + s_2 + s_3$ etc., the sum of the individual elastances. (darafs).
$l = l_1 + l_2 + l_3$ etc., the sum of the individual inductances. (henrys).

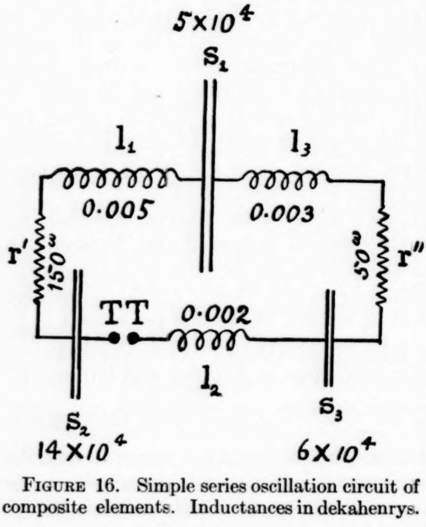


Figure 16. Simple series oscillation circuit of composite elements. Inductances in dekahenrys.

Then if we have only charging oscillations to consider, under the action of a constant impressed emf. E', inserted between terminal TT, we may replace the multiple element system by the equivalent single element system of Figure 1 with resistance r, elastance s, and inductance l.

Discharging Oscillations of Condenser in Simple Series Circuit of Multiple Elements. — Let one of the condensers, say s_1, Figure 16, be initially charged with a quantity Q_1 coulombs to an initial potential $U_0 = Q_1 s_1$ volts, the rest of the system being without charge. Then, after release, the discharge of condenser s_1 will charge the other con-

denser oppositely, in such a manner as to check the oscillation. The oscillations will take place about a condition of voltage equilibrium, such that the emf. of the discharging condenser is equal and opposite to the sum of the emfs. in the other condensers. If the quantity necessary to flow through the circuit in order to attain the condition of emf. equilibrium is q_e coulombs, then

$$(Q_0 - q_e)\, s_1 = q_e\, (s_2 + s_3), \qquad \text{volts} \quad (71)$$

or

$$q_e = Q_0 \frac{s_1}{s_1 + s_2 + s_3} = Q_0 \frac{s_1}{s}. \qquad \text{coulombs} \quad (72)$$

This is the oscillatory part of the charge in s_1. The remainder would reside permanently in the condensers, if there were no dielectric leakage.

Since the passage of q_e coulombs attains the point of equilibrium about which oscillation takes place, the first swing, neglecting damping, carries 2 q_e coulombs through the circuit.

Of the initial stock of energy $W_0 = Q_0^2 s_1/2$ joules, the portion subject to oscillation is, neglecting damping,

$$W_0 \frac{s_1}{s_1 + s_2 + s_3} = W_0 \frac{s_1}{s}, \qquad \text{joules} \quad (73)$$

This portion is alternately electric and magnetic energy. The remainder persists in the electric form, disregarding leakage.

If the impressed voltage E', instead of being applied initially to the component condenser s_1, were applied initially to the equivalent resultant condenser s of Figure 1, whose elastance is the sum of the component elastances, the charge taken by s would be

$$q_0 = Q_0 \frac{s_1}{s}, \qquad \text{coulombs} \quad (74)$$

and the energy of the charge would be

$$w_0 = W_0 \frac{s_1}{s}. \qquad \text{joules} \quad (75)$$

But we have seen that these are precisely the amounts of oscillation-charge and oscillation-energy available in the case of the charged component condenser. Hence we obtain the following rule for the

treatment of composite simple series circuits with component condenser discharges: — Form the equivalent single-element series circuit (Figure 1). Impress the same initial emf. on the single condenser as would be impressed on the component condenser. The discharging oscillations of the single-element system will then be identical with those that would occur in the composite system. After the oscillations have subsided, there will be in the composite system a residual electric energy to take into account, which does not appear in the equivalent single-element system.

Thus in the case of Figure 16, let $s_1 = 5 \times 10^4$, $s_2 = 14 \times 10^4$, $s_3 = 6 \times 10^4$ darafs, $l_1 = 0.05$, $l_2 = 0.02$, $l_3 = 0.03$ henry; $r' = 150$ ohms, $r'' = 50$ ohms, and let an initial charge of 0.02 coulomb be given to s_1 by an impressed emf. of 1000 volts, the other elements being without charge. The initial electric energy of the system $W_0 = 10$ joules. To find the oscillation of the system, we form the equivalent single-element system (Figure 1) with $s = 25 \times 10^4$ darafs, $l = 0.1$ henry, $r = 200$ ohms, and impress 1000 volts initially on the condenser s. This will take a charge of 0.004 coulomb, and an electric energy of 2 joules. These are the oscillation-charge and oscillation-energy of the composite system considered. The oscillations of the system are the same as those indicated in Figures 7 to 15. After the oscillations have subsided, there will be a residual energy of 8 joules in the system, neglecting dielectric leakage, 0.016 coulomb at 800 volts in s_1, —0.004 coulomb at —560 volts, in s_2, and —0.004 coulomb at —240 volts in s_3.

Non-Oscillatory Condenser Discharges.

Although the non-oscillatory discharge of a condenser lies outside of the title of this paper, an outline of the case may be admitted, not only in order to complete the discussion, but also to present therein certain important analogies to the oscillating-current discharge.

If in Figure 12, the semi-circuit resistance ρ is increased until it is equal to z_0, the resistanceless impedance, both the reactance $jl\omega$, the angle ϕ, and the angular velocity ω disappear; so that the discharge becomes non-oscillatory and aperiodic. If ρ is increased beyond this aperiodic value, the discharge continues to be non-oscillatory, but becomes what may be called *ultraperiodic*. We may first consider the ultraperiodic case.

In Figure 17 let op be the exponential time-factor of an ultraperiodic circuit. About op as diameter, construct the semicircle oqp. With center o, and radius $oq = \omega_0$, the resistanceless angular velocity, intersect the semicircle in q. Then the chord pq is the non-oscillatory

or hyperbolic angular velocity Ω of the currents in the circuit.[4] In the case of a circuit resistance $r = 500$ ohms, the hyperbolic angular velocity will be $\Omega = 1936.492$ hyperbolic radians per second, as shown at OPQ, Figure 17.

Analytically, $$\Omega = \sqrt{\iota^2 - \omega_0^2} \quad \text{hyp. radians/sec.} \qquad (76)$$

and the hyperbolic period T of the current and p. d. is $2\,\pi/\Omega$ seconds.

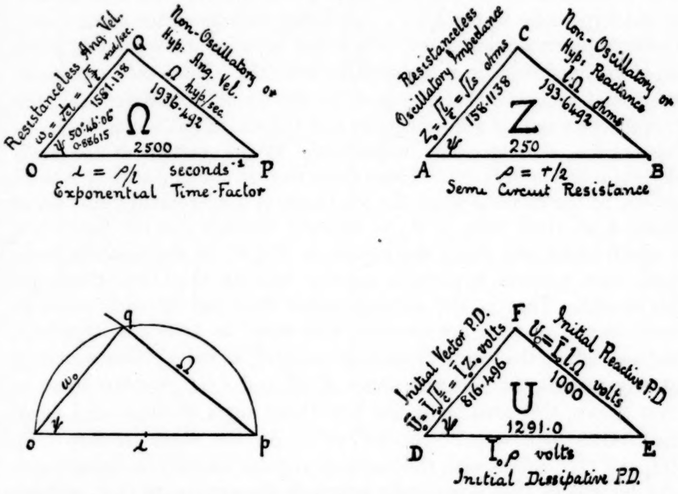


FIGURE 17. Stationary vector-diagrams of hyperbolic angular velocity, impedance, and discharging potential difference in a simple non-oscillating condenser circuit with ultraperiodic resistance.

Similarly, the Z-triangle of Figure 17 has a base $AB = \rho = r/2 = 250$ ohms, and a side AC equal to the resistanceless oscillatory impedance $\sqrt{ls} = 158.1138$ ohms, as in Figures 4 and 12. The remaining side CB opposite to the angle ψ and perpendicular to AC, represents the non-oscillatory or hyperbolic reactance 193.6492 ohms. Operating upon the Z triangle by multiplication with 5.164 amperes, the initial vector-current, we obtain the U triangle DEF, Figure 17.

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[4] For the first publication of the conception of hyperbolic angular velocity of discharges in ultraperiodic circuits, we are indebted to Dr. Alexander Macfarlane. See appended Bibliography.

Figure 18 shows the rotative vector-diagram of voltage and current for the ultraperiodic case considered, and corresponding to Figure 8, the rotative vector-diagram for the oscillating-current case. Lay down the p. d. triangle DEF of Figure 17 at $\bar{I}_0 U_0 O$, Figure 18, to voltage scale as shown. On $O\bar{I}_0$ as semi-axis, construct the rectangular hyperbola $H\bar{E}_0\bar{I}_0\bar{U}_0H'$, whose asymptotes OA and OA' make angles of 45° with $O\bar{I}_0$. From U_0, draw a parallel to $O\bar{I}_0$, intersecting the hyperbola at $\bar{U}_0$. Join $O\bar{U}_0$. Then the angle $\bar{I}_0 O \bar{U}_0 = \psi$ will be the gudermannian of the hyperbolic sector $\bar{I}_0 O \bar{U}_0$; or the hyperbolic sector angle $\bar{I}_0 O \bar{U}_0$ the anti-gudermannian of ψ. From the opposite corresponding point $\bar{E}_0$ of the hyperbola, draw the straight line $O\bar{E}_0$. Then $O\bar{U}_0$ represents the initial vector discharging p. d. in the circuit considered, $O\bar{E}_0$ the initial vector emf. of self-induction, and $O\bar{I}_0$ the initial vector discharging current, corresponding respectively to the vectors of the same denomination in Figure 8. These three vectors, starting at the positions shown, at the moment when the condenser of 4 microfarads, after being charged at 1000 volts p. d., is released through $l = 0.1$ henry and $r = 500$ ohms, run along the hyperbola $H\bar{I}_0H'$, in the positive direction, with uniform hyperbolic angular velocity $\Omega = 1936.492$ hyps. per second. That is, the sectorial areas described by each vector in each second of time are constant and equal to 1936.492 hyperbolic radians, taking the length of the semi-axis $O\bar{I}_0$ as unity. Consequently, at any instant, the sectorial areas $\bar{E}_0 O \bar{I}_0$ and $\bar{I}_0 O \bar{U}_0$ remain equal to that shown, this area being the hyperbolic angle of either, and equal to 1.03172 hyps. $= gd.^{-1}\, 50°.\ 46'.\ 06''$. As the three vectors $O\bar{E}_0$, $O\bar{I}_0$ and $O\bar{U}_0$ rotate with this uniform angular velocity in contact with the hyperbola, they continually approach the asymptote OA', without ever actually reaching it.

The resultant, or vector sum of $O\bar{U}_0$ and $O\bar{E}_0$ is $Or = 2582$ volts, and is equal to the initial vector product of the discharging current $O\bar{I}_0$ and the resistance r of the circuit. This vector also rotates with uniform hyp. angular velocity Ω, over the rectangular hyperbola $h\ r\ h'$, in the direction rh, corresponding to Or, Figure 8. As in Figure 8, the negative of vector Or, or $- \bar{I}_0\, r$, should be drawn in the direction $\bar{I}_0\ O$; but is omitted from the diagram for economy of space. It may be demonstrated that this negative extension of Or, rotating positively over a rectangular hyperbola, the image of $h\ r\ h'$, will always be in vector equilibrium with $O\bar{E}_0$ and $O\bar{U}_0$, so that the geometrical sum of these three vectors at any instant will be zero, just as in the case of Figure 8.

The orthogonal projections of $O\bar{E}_0$, $-Or$, and $O\bar{U}_0$ give the instantaneous values of the discharging p. d., the Ir drop, and the emf

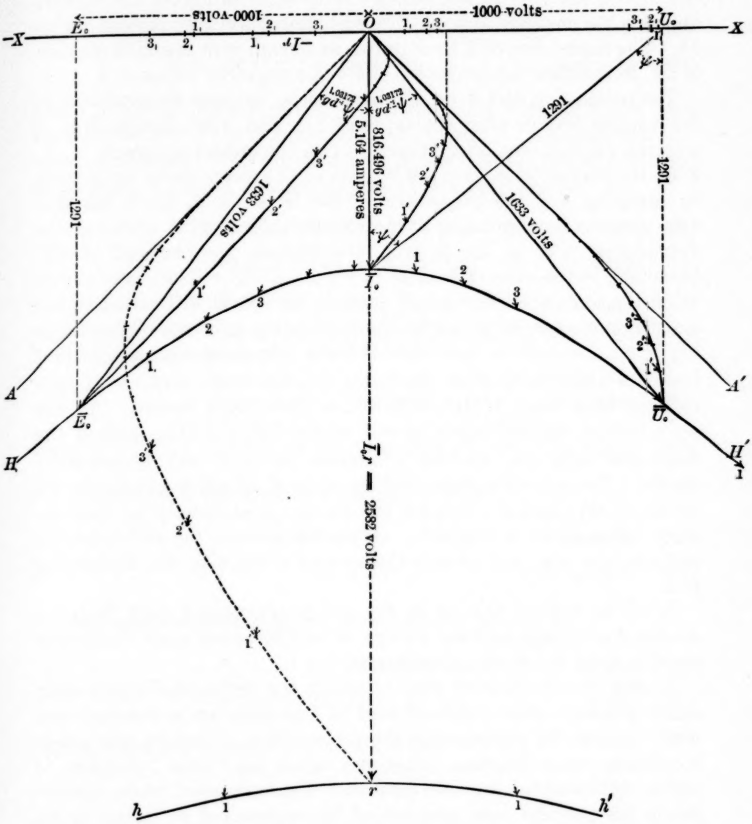

FIGURE 18. Rotative vector-diagram of a non-oscillating current ultra-periodic circuit containing resistance. Instant of release of condenser charge.

of self-induction respectively, after applying the damping factor $\epsilon^{-\Delta t}$. That is, we may take the instantaneous XX projections of these undamped vectors, and then apply the damping factor for the instant

considered; or, we may shrink the vectors by the application of the damping factor before taking projections, as in Figure 8. In the case considered, the damping coefficient $\alpha = 2500$, as in Figure 17; so that applying the damping factor ϵ^{-2500t}, we obtain the curved lines of Figure 18. The dotted line $o3'2'1'r$ is drawn as though with negative rotation of Or, to simulate the projective effect of a negative vector $- I_0 r$.

The points 1, 2, and 3 on the hyperbolas, indicate the positions of the various vectors after the lapse of 1, 2 and 3 ten-thousandths of a second respectively, after release. The corresponding points $1'$, $2'$, $3'$ on the curved lines, give the termini of the same vector as reduced by damping, and the projections of the latter, 1, 2, 3, on the XX axis, give the corresponding instantaneous values in the circuit of the discharging p. d. u, the ir drop, the current, and the emf. of self-induction, in the same manner as in Figure 8. It will be observed that while the undamped vectors all increase in length without limit, the actually projected values under the dominating influence of the damping factor, diminish in time without limit. In particular, the current i reaches a maximum when the vector $O\bar{I}_0$ has swept over 1.0317 hyp. radians, in a time $1.0317/1936.492 = 0.000{,}532{,}8$ second. At the same instant, the self-inductive emf. vector $O\bar{E}_0$ will have reached the transverse axis $O\bar{I}_0$, and its projection on XX will momentarily vanish. Consequently, there will be no emf. of self-induction in the circuit at this instant; because the current is stationary for that instant, being about to diminish. After this instant, the self-inductive emf. changes sign, and propels the current along with the discharging p. d.

It will be noticed that as in the oscillatory-current case, both the discharging voltage and the voltage of self-induction exert dissipative activity upon the discharging current.

It may also be noticed that although the orthogonally-projecting rotating-crank vector-diagrams used in this paper are convenient and useful devices for representing the actions in o. c. circuits, the polar-coordinate vector-diagram, sometimes called the "time" diagram, is not so well adapted for this purpose. The undamped vector quantities in the periodic case may indeed be represented by circles on the polar-coordinate diagram; but the corresponding damped quantities are represented by spirals that are not so easily interpreted as equiangular spirals. Moreover, in the ultraperiodic case, the hyperbolic analogue is missing in the polar-coordinate diagram; so that apparently there is no analogy presented in polar-coordinate representation between the periodic and ultraperiodic cases. It would seem, therefore, that the orthogonally projected "crank-diagram" or "clock-diagram"

has wider applications, in these respects, than the polar-coordinate vector-diagram.

Analytically, we have the following relations : —

The fundamental differential equation for quantity q is satisfied by

$$q = A\epsilon^{-(\iota-\Omega)t} + B\epsilon^{-(\iota+\Omega)t} \qquad \text{coulombs} \quad (77)$$

where A and B are integration constants, while ι and Ω follow from the construction of the triangle OPQ of Figure 17. Choosing the constants consistently with the discharge of a condenser initially charged to potential U_0 volts, the discharging p. d. after t seconds is

$$u = U_0 \cot\psi\, \epsilon^{-\iota t} \sinh(\Omega t + gd^{-1}\psi) \qquad \text{volts} \quad (78)$$

$$= \bar{U}_0 \epsilon^{-\iota t} \sinh(\Omega t + gd^{-1}\psi) \qquad \text{volts} \quad (79)$$

form which q follows by the relation $q = u/s = uc$ coulombs. $\bar{U}_0$ is the initial vector value of the discharging p. d. by Figures 17 and 18.

The instantaneous current i is

$$i = \bar{I}_0 \epsilon^{-\iota t} \sinh \Omega t \qquad \text{amperes} \quad (80)$$

where

$$\bar{I}_0 = U_0 / l\Omega = \bar{U}_0 / z_0. \qquad \text{amperes} \quad (81)$$

The current i will therefore be a maximum when

$$\tanh \Omega t = \Omega\tau = \sin\psi, \qquad \text{numeric} \quad (82)$$

or

$$\Omega t = gd^{-1}\psi. \qquad \text{hyp. radians} \quad (83)$$

The emf. of self-induction in the circuit at any instant is

$$e = U_0 \cot\psi\, \epsilon^{-\iota t} \sinh(\Omega t - gd^{-1}\psi) \qquad \text{volts} \quad (84)$$

$$= \bar{U}_0 \epsilon^{-\iota t} \sinh(\Omega t - gd^{-1}\psi). \qquad \text{volts} \quad (85)$$

The apparent resistance of the circuit u/i is

$$Z = \rho + l\Omega \coth \Omega t. \qquad \text{ohms} \quad (86)$$

That is, the apparent resistance of the circuit, judging from the discharging p. d. and the discharging current, commences at ∞ and tends rapidly to the limit $(\rho + l\Omega)$ ohms.

The instantaneous power of the condenser in the circuit is

$$p = U_0 \bar{I}_0 \cot\psi\, \epsilon^{-2\iota t} \sinh \Omega t \cdot \sinh(\Omega t + gd^{-1}\psi) \qquad \text{watts} \quad (87)$$

$$= \bar{U}_0 \bar{I}_0 \epsilon^{-2\iota t} \sinh \Omega t \cdot \sinh(\Omega t + gd^{-1}\psi). \qquad \text{watts} \quad (88)$$

A complete series of stationary vector-diagrams might be presented following those of Figure 17, and corresponding to those of Figures 7 and 12; but in view of the relative simplicity of formulas (78) to (88), such vector-diagrams have more theoretical than practical interest.

Aperiodic Case.

When ρ, the semi-resistance of the circuit, is just equal to the surge impedance $z_0 = \sqrt{ls}$ of the same; then the circuit is aperiodic, and there is neither a circular angular velocity ω, nor a hyperbolic angular velocity Ω. The aperiodic case may also be represented by a special

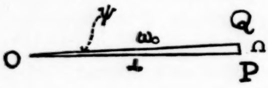

Figure 19. Aperiodic case as limiting condition of ultra-periodic circuit, when $\omega_0 = ℩$. Ω stationary vector-diagram.

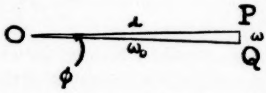

Figure 20. Aperiodic case as limiting condition of o. c. circuit, when $℩ = \omega_0$. ω stationary vector-diagram.

rotating vector-diagram as has been suggested by Macfarlane; but it is easier, for practical purposes, to treat it as a limiting case of the ultra-periodic circuit. We have by (78)

$$u = \bar{U}_0 \cot\psi \, \epsilon^{-℩t} \sinh(\Omega t + gd^{-1}\psi). \qquad \text{volts} \quad (89)$$

Now let Ω become very small, as in Figure 19. Consequently, ψ becomes very small; so that $\cot\psi$ may be replaced by $1/\psi$, and $\sinh(\Omega t + gd^{-1}\psi)$ by $(\Omega t + gd^{-1}\psi)$.

Hence
$$u_{\psi=0} = \frac{U_0}{\psi}\epsilon^{-℩t}(\Omega t + gd^{-1}\psi). \qquad \text{volts} \quad (90)$$

But $\Omega = ℩\psi$ and $gd^{-1}\psi = \psi$ when ψ approaches zero; thus

$$u_{\psi=0} = \frac{U_0}{\psi}\epsilon^{-℩t}(℩\psi t + \psi) = U_0\epsilon^{-℩t}(℩t + 1), \qquad \text{volts} \quad (91)$$

and
$$q_{\psi=0} = Q_0\epsilon^{-℩t}(℩t + 1), \qquad \text{coulombs} \quad (92)$$

$$i_{\psi=0} = -\frac{dq}{dt} = Q_0 ℩^2 t\epsilon^{-℩t}. \qquad \text{amperes} \quad (93)$$

This is a maximum when $t = \tau$; when

$$i_m = Q_0 \iota \epsilon^{-1} = \frac{Q_0}{\tau} \epsilon^{-1}. \qquad \text{amperes} \quad (94)$$

The power in the circuit is

$$p = U_0 Q_0 \iota^2 t \epsilon^{-2\iota t} (\iota t + 1). \qquad \text{watts} \quad (95)$$

We may also derive (91) from the limiting case of the oscillatory discharge. Taking (27), we have

$$u = U_0 \operatorname{cosec} \phi \, \epsilon^{-\iota t} \sin (\omega t + \phi), \qquad \text{volts} \quad (96)$$

and if ω becomes very small, as in Figure 20, the angle ϕ becomes very small; so that $\operatorname{cosec} \phi$ may be replaced by $1/\phi$ and $\sin (\omega t + \phi)$ by $(\omega t + \phi)$. Hence

$$u_{\phi=0} = \frac{U_0}{\phi} \epsilon^{-\iota t} (\omega t + \phi). \qquad \text{volts} \quad (97)$$

But $\omega = \iota\phi$ when ϕ approaches zero; thus,

$$u_{\phi=0} = \frac{U_0}{\phi} \epsilon^{-\iota t} (\iota \phi t + \phi) = U_0 \epsilon^{-\iota t} (\iota t + 1), \qquad \text{volts} \quad (98)$$

as in (91). So that the aperiodic case may be computed either as the limiting oscillatory case with $\omega = 0$, or as the limiting ultraperiodic case with $\Omega = 0$.

Summary of Conclusions.

The orthogonal-projection or rotating-crank vector-diagram of the ordinary a. c. circuit applies also, by extension, to the o. c. circuit.

With the aid of the stationary vector-diagrams, the principal features of any given o. c. case may be simply and speedily deduced.

By making the above stationary vector-diagrams rotative, and subsequently applying the proper damping-factor, the process of oscillation in any given o. c. circuit may be readily visualised.

By interpreting the above diagrams and formulas hyperbolically, the corresponding properties of the ultraperiodic case may be readily computed and visualised. That is, the rotating-crank vector-diagram of the ordinary a. c. circuit applies also, by extension, to the non-oscillatory ultraperiodic condenser circuit.

The properties of the condenser circuit, whether oscillatory or ultra-

periodic, are intimately connected with a certain angle, cos^{-1} (ρ/z) or $cosh^{-1}$ (ρ/z), connected with the circuit constants.

The polar-coordinate type of vector-diagram is not so conveniently adapted to the o. c. circuit as the crank-projection diagram.

The aperiodic case may be treated as the limiting case either of the oscillatory, or of the ultraperiodic, circuit.

List of Symbols Employed.

$A.\ B.$	Constants of integration (coulombs).
a. c.	abbreviation for alternating current.
c, c_1, c_2, c_3	Permittance of a condenser, or of each of several condensers (farads).
γ	Oscillation conductance of an o. c. circuit (mhos).
$\delta_1, \delta_\pi, \delta_{2\pi}$	Logarithmic decrement of an oscillating quantity $\left(\log_\epsilon \frac{V_0}{v}\right)$ during the angular interval of 1 radian, π radians (semi-period), and $2\ \pi$ radians (complete period), of the radius vector (numeric).
ϵ	$= 2.71828 \ldots$
$\bar{E}_0$	Initial vector amplitude of emf. of self-induction (volts $\angle$).
E_0	Initial emf. of self-induction in circuit (volts).
E'	Charging emf. impressed upon a condenser (volts).
$e, \bar{e}$	Instantaneous value of emf. of self-induction in an o. c. circuit, and the undamped value of same (volts).
$\bar{e}_r, e_r$	Undamped and damped instantaneous values of self-induction emf. in phase with the current (volts).
I_0	Initial current strength in a circuit (amperes).
$\bar{I}_0$	Initial vector amplitude oscillating current (amperes $\angle$).
I	$=\bar{I}_0/\sqrt{2}$, the r. m. s. value of initial vector current amplitude (amperes).
$\bar{i}, i$	Undamped and damped instantaneous currents (amperes).
j	$= \sqrt{-1}$, quadrantal operator.
$l, l_1, l_2, l_3,$	Inductance of an o. c. circuit, and of particular parts thereof (henrys).
m	any positive integer (numeric).
n	Frequency of oscillation of a circuit (cycles per second).
o. c.	abbreviation for oscillating-current.
π	$= 3.14159 \ldots$ (numeric)
$Ph.\ Q.$	Any oscillatory physical quantity pertaining to an o. c. circuit.

P_m. $\bar{p}$, p Undamped maximum cyclic power developed by condenser, undamped instantaneous value of same, and damped instantaneous value of same (watts).

$\bar{p}_l$, p_l Undamped and damped instantaneous values of power in inductance (watts).

$\bar{p}_r$, p_r Undamped and damped instantaneous values of power developed by condenser in the resistance of an o. c. circuit (watts).

$p' = ei$ Instantaneous power of the emf. of self-induction (watts).

$2\bar{p}_r$, $2p_r$ Total undamped, and total damped, intantaneous values of power developed in the resistance of an o. c. circuit by condenser and inductance combined (watts).

Q_0. q_0 Initial charge in a condenser (coulombs).

$\bar{Q}_0$ Initial vector amplitude of electric charge in condenser (coulombs $\angle$).

Q $= \bar{Q}_0/\sqrt{2}$, the r. m. s. value of the initial vector amplitude (coulombs).

Q_1 Initial charge of one among several condensers in series (coulombs).

q Instantaneous charge in condenser (coulombs).

q_e Quantity required to flow through an o. c. circuit in order to establish p. d. equilibrium (coulombs).

r' Joulean resistance in an o. c. circuit (ohms).

r'' Hertzian resistance in an o. c. circuit (ohms).

$r = r' + r''$ Total resistance in an o. c. circuit (ohms).

r. m. s. Square root of mean square of an oscillatory quantity.

ρ $= r/2$, Semi total of resistance in an o. c. circuit (ohms).

Φ_0 Total initial magnetic flux linked with a discharging circuit counting all of the turns in the same (volt-seconds).

ϕ Phase angle in an o. c. circuit, and angle of a spiral (radians or degrees).

ψ Phase angle in an ultraperiodic circuit (radians or degrees).

$gd^{-1}\psi$ Antigudermannian of a circular angle, or the hyperbolic angle of which ψ is the gudermannian (hyp. radians).

s, s_1, s_2, s_3, $= 1/c$, elastance of a condenser, or of each of several condensers (darafs).

t Elapsed time from the release of an o. c. system (seconds).

t_1 Time interval (seconds).

T Period of an o. c. circuit (seconds).

τ $= l/\rho$, Oscillation time-constant of an o. c. circuit (seconds).

[illegible]	$= 1/\tau = \rho/l$, time-constant-reciprocal of an o. c. circuit (seconds $^{-1}$).
$\bar{U}_0$	Initial vector amplitude of the discharging p. d. in an o. c. circuit (volts $\angle$).
U	$= \bar{U}_0/\sqrt{2}$, the r. m. s. value of the initial vector discharging p. d. (volts).
U_0	Initial discharging p. d. in a circuit (volts).
$\bar{u}$, u,	Undamped and damped time-values of condenser p. d. in an o. c. circuit (volts).
$\bar{u}_r$, u_r	Undamped and damped time-values of the component of the condenser p. d. in phase with the current (volts).
V_0, V, v	Initial, r. m. s., and instantaneous values of any oscillating quantity, pertaining to an o. c. circuit.
W_0	Initial energy in a condenser or in a reactance (joules).
W_l	Initial vector amplitude of cyclic energy in reactance, *mn*, Figure 7, *MN*, Figures 12 and 14 (joules).
W_m	Undamped maximum cyclic value of energy in a condenser or reactance (joules).
$\bar{w}$, w	Undamped and damped time-values of energy in condenser or reactance (joules).
w_1	Energy dissipated in a circuit in the first energy cycle (joules).
w_c, w_l	Energy in a condenser, and in a reactance, at a specified time (joules).
w_d	Total expenditure of energy by dissipation from a circuit up to t seconds (joules).
w_0	Energy of charge communicated to a series of condensers (joules).
w_s	Semi-system energy at time t (joules).
XX, YY	Cartesian rectangular coordinates.
X_l, X_c	Reactance of an inductance and of a condenser in an o. c. circuit (ohms).
Y	Admittance of an o. c. circuit (mhos).
Z, z	Impedance of an o. c. circuit (ohms).
Z_0, $z_0 = \sqrt{ls}$	Impedance of an o. c. circuit devoid of resistance (ohms).
ω_0	Angular velocity of an o. c. circuit with its resistance ignored (circular radians per second).
ω	Angular velocity of an o. c. circuit in the presence of its actual resistance (circular radians per second).
Ω	Angular velocity of an ultraperiodic circuit (hyperbolic radians per second).

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